THE SEPTEMBER MEETING IN MADISON

The forty-fifth summer meeting and twenty-first colloquium of the American Mathematical Society were held at the University of Wisconsin, Madison, Wisconsin, from Tuesday to Friday, September 5–8, 1939, preceded by the summer meeting of the Mathematical Association of America.

The attendance was about three hundred fifty including the following two hundred forty-two members of the Society:

R. P. Agnew, Leonidas Alaoglu, A. A. Albert, G. E. Albert, O. W. Albert, F. E. Allen, R. C. Archibald, H. A. Arnold, J. V. Atanasoff, W. L. Ayres, Reinhold Baer, F. E. Baker, R. H. Bardell, R. C. F. Bartels, J. H. Bell, P. O. Bell, A. A. Bennett, D. L. Bernstein, S. F. Bibb, G. A. Bliss, B. W. Brewer, R. W. Brink, F. L. Brooks, R. E. Bruce, H. E. Buchanan, J. H. Bushey, Jewell H. Bushey, W. H. Bussey, W. D. Cairns, H. H. Campaigne, M. E. Carlen, I. S. Carroll, W. B. Carver, C. R. Cassity, Harold Chatland, E. W. Chittenden, A. H. Clifford, J. B. Coleman, E. G. H. Comfort, L. P. Copeland, J. J. Corliss, Richard Courant, H. S. M. Coxeter, A. R. Crathorne, L. L. Cronvich, E. L. Crow, D. R. Curtiss, H. T. Davis, W. M. Davis, F. F. Decker, L. A. V. DeCleene, C. H. Denbow, Douglas Derry, L. L. Dines, B. F. Dostal, R. H. Downing, Arnold Dresden, R. J. Duffin, J. M. Earl, M. C. Eide, Churchill Eisenhart, Benjamin Epstein, G. C. Evans, G. W. Evans, H. P. Evans, C. J. Everett, L. R. Ford, Tomlinson Fort, J. S. Frame, Bernard Friedman, T. C. Fry, M. G. Gaba, E. R. Garbe, B. E. Gatewood, B. P. Gill, W. O. Gordon, W. C. Graustein, L. M. Graves, V. G. Grove, B. L. Hagen, D. W. Hall, O. G. Harrold, W. L. Hart, W. W. Hart, Camilla Hayden, E. R. Hedrick, E. D. Hellinger, Olaf Helmer, A. D. Hestenes, M. R. Hestenes, Einar Hille, J. J. L. Hinrichsen, D. L. Holl, T. R. Hollcroft, E. M. Hull, M. G. Humphreys, W. A. Hurwitz, D. H. Hyers, M. H. Ingraham, Dunham Jackson, S. B. Jackson, Nathan Jacobson, R. L. Jeffrey, Walter Jennings, C. M. Jensen, Fritz John, M. M. Johnson, E. R. van Kampen, Wilfred Kaplan, A. J. Kempner, R. B. Kershner, P. W. Ketchum, H. R. Kingston, E. E. Knight, Fulton Koehler, R. E. Langer, C. G. Latimer, V. V. Latshaw, D. H. Lehmer, R. A. Leibler, Walter Leighton, C. A. Lester, G. H. Ling, Dorothy McCoy, N. H. McCoy, W. C. McDaniel, J. V. McKelvey, Brockway McMillan, E. J. McShane, C. C. MacDuffee, Saunders MacLane, H. F. MacNeish, Ralph Mansfield, H. W. March, Morris Marden, W. T. Martin, L. C. Mathewson, A. E. May, J. R. Mayor, L. E. Mehlenbacher, D. D. Miller, W. L. G. Mitchell, Virginia Modesitt, W. L. Moore, C. W. Moran, L. J. Mordell, Vladimir Morkovin, D. S. Morse, Marston Morse, David Moskovitz, E. J. Moulton, H. T. Muhly, J. L. Nagel, Tadasi Nakayama, C. J. Nesbitt, Albert Neuhaus, E. N. Oberg, Rufus Oldenburger, L. F. Ollmann, Isaac Opatowski, Alexander Oppenheim, F. W. Owens, H. B. Owens, J. C. Oxtoby, G. A. Parkinson, E. W. Paxson, G. H. Peebles, W. H. Pell, I. E. Perlin, Sam Perlis, G. W. Petrie, H. P. Pettit, Everett Pitcher, G. B. Price, J. W. Querry, Tibor Radó, W. T. Reid, R. W. Rempfer, R. G. D. Richardson, R. F. Rinehart, G. de B. Robinson, S. G. Roth, W. E. Roth, L. L. Runge, M. G. Scherberg, O. F. G. Schilling, E. R. Schneckenburger, G. E. Schweigert, W. T. Scott, C. E. Sealander, Hyman Serbin, C. G. Shover, R. G. Simond, M. E. Sinclair, E. R. Sleight, M. F. Smiley, C. E. Smith, E. R. Smith, G. W. Smith,

Virgil Snyder, E. S. Sokolnikoff, I. S. Sokolnikoff, N. E. Steenrod, B. M. Stewart, M. H. Stone, E. B. Stouffer, M. M. Sullivan, Otto Szász, Gabor Szegö, M. E. Taylor, W. C. Taylor, B. J. Tepletsky, R. M. Thrall, Leonard Tornheim, H. C. Trimble, P. L. Trump, A. W. Tucker, H. L. Turrittin, Henry Van Engen, E. B. Van Vleck, J. H. Van Vleck, V. J. Varino, J. I. Vass, Oswald Veblen, G. E. Wahlin, R. J. Walker, A. D. Wallace, J. A. Ward, J. H. Weaver, M. S. Webster, M. J. Weiss, E. T. Welmers G. W. Whaples, P. M. Whitman, G. T. Whyburn, W. M. Whyburn, Norbert Wiener, L. R. Wilcox, L. A. Wolf, M. C. Wolf, F. E. Wood, H. A. Wood.

Many of the members remembered with pleasure the two previous summer meetings of the Society in Madison in 1913 and 1927, and as on the two earlier occasions the University of Wisconsin proved a perfect host. The growth of the Society is reflected in the records of these three meetings. In 1913 there were 57 members present and 46 papers were presented; in 1927, 133 members and 69 papers; while at this meeting there were 242 members and 120 papers, the largest number of papers ever presented at any one meeting of the Society.

The program was featured by two sets of Colloquium Lectures. The lectures on *Structure of algebras* were delivered by Professor A. A. Albert of the University of Chicago on Tuesday afternoon, Wednesday, Thursday, and Friday mornings. The lectures on *Convex bodies* were delivered by Professor M. H. Stone of Harvard University on Wednesday morning, Thursday afternoon, and Friday morning. Professor Albert's lectures appeared as volume 24 of the Colloquium Publications at the time of the meeting and Professor Stone's lectures will appear as a volume in this series at some future date. This was the third Madison Colloquium, the previous ones being given by Professors L. E. Dickson and W. F. Osgood in 1913, and by Professors E. T. Bell and Anna Pell Wheeler in 1927.

President G. C. Evans presided at each initial Colloquium Lecture and at the general session Wednesday morning. Vice-President R. E. Langer presided at the general session Thursday afternoon. The presiding officers at the sectional sessions were as follows: Analysis, Professors E. J. McShane, H. E. Buchanan, and R. P. Agnew; Algebra, Professors D. H. Lehmer, and C. C. MacDuffee; Algebra and Geometry, Professors Virgil Snyder and H. S. M. Coxeter; Topology, Professor Tibor Radó.

The dormitories of the University were opened to the members of both organizations and their guests, and headquarters for the meetings were at the Gatehouse.

Arrangements for the entertainment of those attending the meetings were excellent. Swimming and boating were available on Lake Mendota at all times, and arrangements were made for golf and tennis. On Monday and Tuesday afternoons the ladies of the local De-

partment of Mathematics served tea in the Pine Room of the refectory. On Wednesday afternoon there were excursions to the Banta Publishing Company, the United States Forest Products Laboratory, and the Yerkes Observatory. In addition on Wednesday there was a trip around Lake Mendota by launch and a picnic on the lake shore at Picnic Point.

A joint dinner for members of the Society and Association and their guests was held at the Loraine Hotel on Thursday evening, President G. C. Evans presiding as toastmaster. Dean G. C. Sellery of the University of Wisconsin welcomed the mathematicians on behalf of the University. Professor W. C. Graustein spoke of the unfortunate postponement of the International Congress. Professor Oswald Veblen and Dr. T. C. Fry told of the plans for the new Mathematical Reviews and participated in the ensuing lively discussion. Professor W. D. Cairns spoke of amusing and interesting experiences as Secretary of the Association. At the close of the dinner Dean E. B. Stouffer presented a joint resolution on behalf of both mathematical organizations thanking the President and officers and the members of the Department of Mathematics of the University of Wisconsin, and the officials of the Banta Publishing Company for their competent arrangements and cordial hospitality. The attendance at the dinner was two hundred fifty-one.

The Council met on Wednesday, September 6, at 8:15 P.M. in the small dining room of the refectory. An adjourned meeting was held in the Loraine Hotel after the dinner on the evening of September 7.

The Secretary announced the election of the following thirty persons to membership in the Society:

Professor Olin Blair Ader, Wofford College;
Dr. Burns W. Brewer, Texas Agricultural and Mechanical College;
Professor Charles Edwin Bures, The College of Idaho, Caldwell, Idaho;
Dr. Nathaniel Coburn, University of Texas;
Mr. Robert Coleman, Jr., Wilberforce University, Wilberforce, Ohio;
Mr. Thomas Carlson Doyle, Princeton University;
Mr. William Frederick Eberlein, Shawano, Wis.;
Miss Laura Christine Efird, Hugh Morson High School, Raleigh, N. C.;
Mr. Raymundo Acosta Favila, Manila, P. I.;
Dr. Floyd Grayston Fisher, University of Santa Clara;
Professor John Steiner Gold, Bucknell University;
Dr. Ernst D. Hellinger, Northwestern University;
Mr. John S. Hickman, Rochester Junior College, Rochester, Minn.;
Dr. Gabriel Horvay, New York, N. Y.;
Dr. Hsien-Yü-Hsü, Washington University, St. Louis, Mo.;
Mr. Fulton Koehler, University of Minnesota;
Mr. Ogden Lillard, New York, N. Y.;

Mr. John Lovell Loughborough, Lincoln Institute, Hollywood, Calif.;
Sister Mary Michael Maloney, D'Youville College, Buffalo, N. Y.;
Mr. F. C. W. Olson, American Can Company, Maywood, Ill.;
Professor Edward Monroe Joseph Pease, Rhode Island State College;
Professor Joseph Alphonso Pierce, Atlanta University, Atlanta, Ga.;
Dr. Jenny Eugenie Rosenthal, Brooklyn College;
Dr. Michael A. Sadowsky, Armour Institute of Technology;
Mr. Charles Louis Seebeck, Jr., University of North Carolina;
Mr. Robert Richmond Singleton, Princeton, N. J.;
Dr. Wolfgang Sternberg, New York, N. Y.;
Mr. Duane Morton Studley, Colorado Springs, Colo.;
Mr. Benjamin Joseph Trapani, Pennsylvania State College;
Dr. John William Theodore Youngs, Ohio State University.

The following two persons were admitted to the Society in accordance with the reciprocity agreement with the London Mathematical Society:

Professor W. H. McCrea, Queen's University, Belfast, Ireland;
Professor Frederick Riesz, University of Szeged, Szeged, Hungary.

The following appointments by President Evans were reported: as representative of the Society at the Sesquicentennial Celebration of Georgetown College (now Georgetown University) on May 28–June 3, Professor A. E. Landry; as representative of the Society at the celebration of the fiftieth anniversary of the founding of the University of New Mexico on June 4–5, Professor C. V. Newsom; as representative of the Society at the celebration of the fiftieth anniversary of the founding of the Catholic University of America on November 11–13, Professor F. D. Murnaghan; to replace Professor C. R. Adams on the Committee to consider freer methods of nominating and electing members-at-large in the Council, Professor L. R. Ford; as a representative of the Society on the Council of the American Association for the Advancement of Science for 1939, Professor L. L. Dines.

Meetings of the Society were set as follows: at the California Institute of Technology on December 2, 1939, and in Chicago on April 12–13, 1940.

Professor Oswald Veblen (chairman) and Drs. T. C. Fry and Warren Weaver were continued in charge of the new journal, Mathematical Reviews. An article concerning this new development of the Society's work appeared in the September number of this Bulletin.

In view of the distressful world situation, the International Congress of Mathematicians which was to be held at Cambridge, Massachusetts, in September, 1940, was postponed. An Emergency Executive Committee, consisting of Professors G. D. Birkhoff, W. C. Graustein, Einar Hille, M. H. Ingraham, J. R. Kline, Marston Morse,

R. G. D. Richardson, and M. H. Stone, was appointed to act during the interim.

A revision of the by-laws of the Society to facilitate the handling of the financial business which has grown to such considerable proportions was approved and this revision will be submitted to the Society at the October meeting.

A committee on the reorganization of the business of the Secretariat was appointed, consisting of Professors L. M. Graves (chairman), W. B. Fite, J. R. Kline, C. C. MacDuffee, E. J. McShane, W. T. Martin, and C. B. Morrey.

Professor Tomlinson Fort for the Editorial Committee on the Bulletin announced that the edition beginning with the September 1939 issue will be increased by one hundred copies.

Professor C. N. Moore was reappointed as representative of the Society on the board of the American Year Book for a period of three years.

Professors Oystein Ore and G. T. Whyburn were appointed Colloquium Lecturers for 1941 with the understanding that one or both may be moved forward to 1940. It was announced that Volumes 23 and 24 of the Colloquium series by Professors Gabor Szegö and A. A. Albert were off the press.

The titles and cross-references to abstracts of papers read at this meeting are given below. Papers numbered 1 to 11 were presented Tuesday afternoon; papers 12 to 14, Wednesday morning; papers 15 to 40, Thursday morning; papers 41 to 44, Thursday afternoon; and papers 45 to 54, Friday morning. Papers 55 to 120, whose abstract numbers are followed by the letter *t*, were read by title. Paper number 1 was read by Professor Moskovitz, 30 by Dr. Kershner, 40 by Dr. Hall, 45 by Dr. Scott, 46 by Professor Martin, and 48 by Dr. Duffin. Dr. Andrew Sobczyk was introduced by Professor H. F. Bohnenblust, Mr. Bernard Dimsdale by Professor DunhamJackson, Professors A. E. Ross and W. S. Kimball by Professor W. L. Ayres, and Dr. I. Malkin by Professor W. T. Martin.

1. David Moskovitz and L. L. Dines: *On the supporting-plane property of a convex body.* (Abstract 45-9-351.)

2. M. F. Smiley: *A note on measure functions in a lattice.* (Abstract 45-7-278.)

3. D. H. Hyers: *A generalization of Fréchet's differential.* (Abstract 45-9-325.)

4. Andrew Sobczyk: *Projections in Minkowski and Banach spaces.* (Abstract 45-9-378.)

5. Wilfred Kaplan: *Regular curve-families filling the plane.* (Abstract 45-9-328.)

6. Isaac Opatowski: *Differential equations containing functionals of unknown functions.* (Abstract 45-9-357.)

7. H. A. Arnold: *Defective groups.* (Abstract 45-9-296.)

8. R. M. Thrall: *A note on groups of class 3.* (Abstract 45-9-381.)

9. Reinhold Baer: *Nilpotent groups and their generalizations.* (Abstract 45-9-299.)

10. P. M. Whitman: *Free lattices.* (Abstract 45-9-390.)

11. G. W. Whaples: *On the structure of moduls with a commutative algebra as operator domain.* Preliminary report. (Abstract 45-9-389.)

12. L. R. Wilcox: *A new foundation for projective differential geometry.* (Abstract 45-9-393.)

13. E. J. McShane: *Generalized curves.* (Abstract 45-9-342.)

14. Everett Pitcher: *Critical points of a map to a circle.* (Abstract 45-9-362.)

15. B. E. Gatewood: *Thermal stresses in long cylindrical bodies.* (Abstract 45-9-314.)

16. R. E. Langer: *On the stability of the laminar flow of a viscous fluid.* (Abstract 45-9-332.)

17. Dunham Jackson: *Orthogonal polynomials with auxiliary conditions.* (Abstract 45-9-326.)

18. Bernard Dimsdale: *Degree of approximation by linear combinations of powers.* (Abstract 45-9-309.)

19. G. H. Peebles: *On equivalence of certain types of series of orthonormal functions.* (Abstract 45-9-358.)

20. E. N. Oberg: *A note on a certain phase of best approximation.* (Abstract 45-9-354.)

21. M. S. Webster: *Note on certain Lagrange interpolation polynomials.* (Abstract 45-9-388.)

22. Bernard Friedman: *Fourier coefficients of bounded functions.* (Abstract 45-9-313.)

23. L. E. Mehlenbacher: *Determination of the asymptotic behavior of the solutions of differential equations of the Fuchsian type; the case of $n+2$ regular singular points.* (Abstract 45-7-282.)

24. H. S. M. Coxeter: *A new form of coordinates for the polytope 2_{21} whose twenty-seven vertices correspond to the lines on the general cubic surface.* (Abstract 45-9-308.)

25. T. R. Hollcroft: *Anomalous systems of primals.* (Abstract 45-9-324.)

26. P. O. Bell: *Projective invariants of a curve on a surface.* (Abstract 45-9-301.)

27. H. T. Muhly: *Valuations and infinitely near algebraic loci.* II. Preliminary report. (Abstract 45-9-352.)

28. C. G. Latimer: *A class of diophantine equations.* (Abstract 45-9-333.)

29. D. H. Lehmer: *Certain congruential properties of spherical harmonics.* (Abstract 45-9-334.)

30. Philip Hartman and R. B. Kershner: *On some limit theorems for number theoretic functions.* (Abstract 45-9-322.)

31. A. E. Ross: *A note on representation of integers by positive quaternary quadratic forms.* (Abstract 45-9-369.)

32. H. H. Campaigne: *Suschkewitsch correspondences.* (Abstract 45-9-307.)

33. L. F. Ollmann: *On cubic ring graphs.* (Abstract 45-9-356.)

34. W. L. Ayres: *Peano spaces as the theory of continuous images of intervals.* (Abstract 45-9-297.)

35. A. D. Wallace: *On quasi-monotone transformations.* (Abstract 45-9-387.)

36. G. T. Whyburn: *Non-alternating interior retracting transformations.* (Abstract 45-7-281.)

37. N. E. Steenrod: *Homology groups of the complement of a closed set in an n-sphere.* (Abstract 45-9-379.)

38. Edith R. Schneckenburger: *On 1-bounding monotonic transformations which are equivalent to homeomorphisms.* (Abstract 45-9-372.)

39. O. G. Harrold: *Topological properties of rectifiable continua.* (Abstract 45-9-321.)

40. D. W. Hall (National Research Fellow) and G. T. Whyburn: *An analysis of arc-preserving transformations.* (Abstract 45-11-399.)

41. Fritz John: *Special solutions of certain difference equations.* (Abstract 45-9-327.)

42. W. T. Reid: *Sufficient conditions by expansion methods for multiple integral problems of the calculus of variations.* (Abstract 45-9-366.)

43. Olaf Helmer: *The principal ideal theorem in domains of integral functions.* (Abstract 45-9-323.)

44. R. P. Agnew: *On kernels of faltung transformations.* (Abstract 45-9-293.)

45. W. T. Scott and H. S. Wall: *A convergence theorem for continued fractions.* (Abstract 45-7-277.)

46. Stefan Bergmann and W. T. Martin: *On the existence of a function of integrable square which satisfies an infinite system of integral relations.* (Abstract 45-9-302.)

47. Benjamin Epstein: *Growth properties of analytic functions of two complex variables.* (Abstract 45-9-311.)

48. A. C. Schaeffer and R. J. Duffin: *A refinement of the inequality of the brothers Markoff.* (Abstract 45-9-371.)

49. I. E. Perlin: *Indefinitely differentiable functions with prescribed least upper bounds.* (Abstract 45-9-359.)

50. Margarete C. Wolf: *Transformations of bases for relative sets over a non-commutative field.* (Abstract 45-9-394.)

51. Albert Neuhaus: *On products of normal semi-fields.* (Abstract 45-9-353.)

52. N. H. McCoy: *A generalization of Ostrowski's theorem on matric identities.* (Abstract 45-9-339.)

53. Rufus Oldenburger: *Infinite powers of matrices and characteristic roots.* (Abstract 45-9-355.)

54. W. E. Roth: *On the unilateral equation in matrices.* II. (Abstract 45-9-370.)

55. Ingo Maddaus: *On types of "weak" convergence in normed vector spaces.* Preliminary report. (Abstract 45-9-337-*t*.)

56. Richard Brauer: *On groups whose order is divisible by the first power of a prime.* II. Preliminary report. (Abstract 45-9-304-*t*.)

57. Deane Montgomery and Leo Zippin: *Topological transformation groups.* (Abstract 45-9-348-*t*.)

58. P. R. Halmos: *On a necessary condition for the strong law of large numbers.* (Abstract 45-9-320-*t*.)

59. H. A. Simmons and J. F. Paydon: *On the shortest time of fall from a curve to a point.* (Abstract 45-9-377-*t*.)

60. D. W. Hall (National Research Fellow) and J. L. Kelley: *On continuous collections of orbits.* (Abstract 45-9-319-*t*.)

61. W. J. Trjitzinsky: *General theory of singular integral equations with real kernels.* (Abstract 45-9-382-*t*.)

62. W. S. Kimball: *Partial derivatives of derivatives.* (Abstract 45-9-330-*t*.)

63. Garrett Birkhoff and John Dyer-Bennet: *A note on real algebraic functions.* (Abstract 45-5-162-*t*.)

64. Garrett Birkhoff: *Lattice theory of Browerian logic.* (Abstract 45-7-260-*t*.)

65. Morgan Ward: *An arithmetical characterization of a modular lattice.* (Abstract 45-7-280-*t*.)

66. J. F. Ritt: *On intersections of algebraic differential manifolds.* (Abstract 45-7-274-*t*.)

67. J. F. Ritt and E. R. Kolchin: *On certain ideals of differential polynomials.* (Abstract 45-7-275-*t*.)

68. E. R. Lorch: *Means of iterated transformations in reflexive vector spaces.* (Abstract 45-7-270-*t*.)

69. R. P. Boas (National Research Fellow): *Integral functions bounded on the real axis.* (Abstract 45-7-261-*t*.)

70. R. H. Cameron: *Extensions of Wiener's general Tauberian theorem.* (Abstract 45-7-262-*t*.)

71. Oystein Ore and J. E. Eaton: *Remarks on multigroups.* (Abstract 45-7-273-*t*.)

72. J. J. De Cicco: *The differential geometry of the curves of the Kasner plane.* (Abstract 45-7-264-*t*.)

73. J. J. De Cicco: *The conics of the Kasner plane.* (Abstract 45-7-265-*t*.)

74. R. P. Dilworth: *A characterization of complemented modular lattices.* (Abstract 45-7-266-*t*.)

75. R. P. Dilworth: *Note on the prime elements of a modular lattice.* (Abstract 45-7-267-*t*.)

76. Morgan Ward: *The ideal operators of a lattice.* (Abstract 45-7-289-*t*.)

77. Barkley Rosser: *A new lower bound for the exponent in the first case of Fermat's last theorem.* (Abstract 45-7-283-*t*.)

78. W. T. Scott and H. S. Wall: *A convergence theorem for continued fractions.* II. (Abstract 45-7-285-*t*.)

79. Walter Leighton: *Proper continued fractions.* (Abstract 45-9-290-*t*.)

80. G. T. Whyburn: *On the interiority of real functions.* (Abstract 45-9-392-*t*.)

81. G. T. Whyburn: *A relation between interior and non-alternating transformations.* (Abstract 45-9-391-*t*.)

82. M. S. Knebelman: *Scalar invariants of the vector vortex excess.* (Abstract 45-9-331-*t*.)

83. W. T. Puckett: *On regular transformations.* (Abstract 45-9-364-*t*.)

84. J. F. Ritt and H. W. Raudenbush: *Ideal theory and algebraic difference equations.* (Abstract 45-9-367-*t*.)

85. Garrett Birkhoff: *Neutral elements in general lattices.* (Abstract 45-9-303-*t*.)

86. D. C. Lewis: *The separability of Pfaffian transformations.* (Abstract 45-9-336-*t*.)

87. C. W. Vickery: *On two different types of linear continua.* (Abstract 45-9-385-*t*.)

88. C. W. Vickery: *On spaces* (V_{ω_α}) *and spaces* Λ_{ω_α}. (Abstract 45-9-384-*t*.)

89. C. W. Vickery: *Types of limit points in spaces (V).* (Abstract 45-9-386-*t.*)

90. C. W. Vickery: *A theorem on abstract sets.* (Abstract 45-9-383-*t.*)

91. Edward Kasner and J. J. De Cicco: *Trihornometry of second order.* (Abstract 45-9-329-*t.*)

92. Jakob Levitzki: *On multiplicative systems.* (Abstract 45-9-335-*t.*)

93. L. M. Graves: *Some general approximation theorems.* (Abstract 45-9-317-*t.*)

94. L. M. Graves: *On the fundamental lemma of Haar in the calculus of variations.* (Abstract 45-9-316-*t.*)

95. J. W. T. Youngs: *Arc-spaces.* (Abstract 45-9-395-*t.*)

96. J. W. T. Youngs: *Generalized cyclic elements.* (Abstract 45-9-396-*t.*)

97. Reinhold Baer: *A Galois theory of linear systems over commutative fields.* (Abstract 45-9-298-*t.*)

98. Deane Montgomery and Leo Zippin: *Transformation groups characterizing certain three-manifolds.* (Abstract 45-9-349-*t.*)

99. C. B. Morrey: *The problem of Plateau on a Riemannian manifold.* Preliminary report. (Abstract 45-9-350-*t.*)

100. D. W. Hall (National Research Fellow): *On pointwise periodic homeomorphisms in the plane.* (Abstract 45-9-318-*t.*)

101. E. J. McShane: *Necessary conditions for a minimum in generalized-curve problems of the calculus of variations.* (Abstract 45-9-343-*t.*)

102. E. J. McShane: *Existence theorems for Bolza problems in the calculus of variations.* (Abstract 45-9-344-*t.*)

103. E. J. McShane: *An estimate for the Weierstrass $\mathcal{E}$-function.* (Abstract 45-9-340-*t.*)

104. E. J. McShane: *A remark concerning sufficiency theorems for the problem of Bolza.* (Abstract 45-9-341-*t.*)

105. I. Malkin: *On the elastic problem of the plate.* Preliminary report. (Abstract 45-9-338-*t.*)

106. M. S. Robertson: *Typically-real functions with $a_n=0$ for $n\equiv 0$* mod 4. (Abstract 45-9-368-*t.*)

107. W. T. Reid: *A theorem on continuous functions in abstract spaces.* (Abstract 45-9-365-*t.*)

108. I. J. Schoenberg: *On local convexity in euclidean spaces.* (Abstract 45-9-373-*t.*)

109. Everett Pitcher: *Identification of two subsets.* (Abstract 45-9-363-*t.*)

110. B. J. Pettis: *Absolutely continuous functions in vector spaces.* (Abstract 45-9-360-*t*.)

111. B. J. Pettis: *The uniform differentiation of families of numerical functions.* (Abstract 45-9-361-*t*.)

112. R. P. Agnew: *On translations of functions and sets.* (Abstract 45-9-294-*t*.)

113. F. A. Ficken: *Cones and vector spaces.* (Abstract 45-9-312-*t*.)

114. R. S. Burington: *On transient similarity and equivalence in linear networks.* (Abstract 45-9-305-*t*.)

115. A. D. Michal: *Fréchet differentials and M-differentials in Banach spaces.* (Abstract 45-9-346-*t*.)

116. A. D. Michal: *A differential in linear topological spaces.* (Abstract 45-9-345-*t*.)

117. A. D. Michal, R. Davis, and Max Wyman: *Polygenic functions in general analysis.* (Abstract 45-9-347-*t*.)

118. Warren Ambrose: *Some properties of measurable stochastic processes.* (Abstract 45-9-295-*t*.)

119. Gabor Szegö: *On the gradient of solid harmonic polynomials.* (Abstract 45-11-403-*t*.)

120. Ben Dushnik and E. W. Miller: *Concerning similarity transformations of linearly ordered sets.* (Abstract 45-11-397-*t*.)

W. L. AYRES,
Associate Secretary

SHORTER NOTICES

Die Gegenwärtige Lage in der mathematischen Grundlagenforschung. Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie. By Gerhard Gentzen. (Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, new series, no. 4.) Leipzig, Hirzel, 1938. 44 pp.

This is really two books printed in one volume. The first one, "Die gegenwärtige Lage in der mathematischen Grundlagenforschung," has also appeared in Deutsche Mathematik, vol. 3 (1938), pp. 255–268. The second one, "Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie," is a revision of Gentzen's famous paper in the Mathematische Annalen, vol. 112 (1936), pp. 493–565.

The first book is a well written summary of the present status of foundations, and contains one of the most lucid accounts of the Brouwer viewpoint that the present reviewer has seen. The distinction between the Brouwer and Hilbert schools is presented from the point of view of their treatment of the infinite. For Brouwer, who always insists on finite constructibility, the infinite exists only in the sense that he can at any time take a larger (finite) set than any which he has taken hitherto. Hilbert would treat of infinite sets by the same methods used for finite sets, as if he could comprehend them in their entirety. Gentzen refers to this point of view as the "as if" point of view. He presents various paradoxes which arise when the "as if" method is used without proper care.

This of course opens the question of what is "proper care." In the nature of things, the Brouwer method must fail to produce a paradox, since it never leaves the domain of the constructive finite. However the Brouwer method does not produce sufficient mathematical theory for physical and engineering uses. So Brouwer's method must be described as "excessive care."

A proposed way out of the difficulty is to base the "as if" method on an appropriate formal system, and use the Brouwer method to prove that the formal system is without a contradiction. For none of the various formal systems so far proposed has such a proof of freedom from contradiction been given. More serious still, a well known theorem of Gödel says that if a logic L_1 is used to prove the freedom from contradiction of a logic L_2, then L_1 must in some respects be stronger than L_2. So the above program will fall through unless one can point out some respect in which the Brouwer method is stronger than the "as if" method. Gentzen thinks he has found it.

His idea is to use the Brouwer method, involving the use of transfinite induction up to a certain ordinal α, to prove the freedom from contradiction of that part of the "as if" method which involves transfinite induction only up to an appropriate smaller ordinal β. If β is fairly large, the resulting "as if" method, though restricted, should be adequate for physics and engineering.

In the second book, Gentzen illustrates the above proposal by using the Brouwer method, with induction up to ϵ_0, to prove the freedom from contradiction of number theory with induction up to any ordinal less than ϵ_0. An important gap in the proof is the absence of a *constructive* proof that induction is valid up to ϵ_0. Gentzen himself comments on this gap, and expresses the belief that it will shortly be filled.

The present proof of freedom from contradiction is made considerably simpler than the earlier proof (in Mathematische Annalen—see above) by using Gentzen's LK-calculus, rather than his NK-calculus.

The proof is too complicated to be sketched here. However it is worth saying that what Gentzen does is to describe a means of attaching an ordinal number (less than ϵ_0) to any proof of number theory. He then describes how, if one had a proof of a contradiction, one could find a second proof of a contradiction having a smaller ordinal number than the first proof.

J. BARKLEY ROSSER

Reports of a Mathematical Colloquium. Series 2, no. 1. Edited by Karl Menger. Notre Dame University Press, 1939. 64 pp.

This booklet of seven papers begins a continuation of the earlier series of reports issued from 1928 to 1936 by the Vienna Colloquium under the leadership of Professor Menger.

"Stability of Limited Competition and Cooperation," by G. C. Evans and Kenneth May, deals with two producers. Under simplifying assumptions, conditions are found on the coefficients of the demand and cost functions in order that an equilibrium point be possible, that this point be competitive or cooperative, and that the equilibrium be stable. Under strong hypotheses, similar methods are applied to labor, leading to the conclusion that a union able to control the labor supply for a given industry can make the introduction of machinery unprofitable to the entrepreneur.

"On Linear Sets in Metric Spaces," by Karl Menger and Arthur Milgram, contains four theorems, of which a special consequence is the known theorem that, in a complete and convex metric space, any two distinct points are joined by a subset congruent to a segment of the euclidean line.

The third paper is "Partially Ordered Sets, Separating Systems and Inductiveness," by A. N. Milgram. It is unfortunate that this interesting and substantial paper gives the impression of having been written hurriedly and printed without proofreading. The results given seem to be new and significant.

Essentially, certain portions of the Dedekind theory of the continuum are so adjusted as to be useful in studying partially ordered sets. Let A be a partially ordered set. A subset L is called a lower section of A if $a \in L$ and $b < a$ imply $b \in L$; if B is a subset of A, the set of those elements a of A with the property that $a \leqq b$ for every b in B is called the under section of B. In terms of these concepts, it is found possible to define well-ordered subsets and to associate with each element of A a unique subset which is well-ordered and has other useful properties; this association is produced once with and once without an application of transfinite induction. Separation—analogous to that effected in the continuum by the rational numbers—is defined intrinsically and extrinsically, the equivalence of the definitions is proved, and the powers of well-ordered subsets are compared with the powers of systems of separating sets. If A has a denumerable separating system, it is shown that A can be mapped on an interval of the continuum in such a way that order-relations are preserved. Several applications are given, chiefly to problems in topology.

"Postulates for the Ratio of Division," by B. J. Topel, deals with a set of elements and a real-valued function f of trios of these elements. f is subject to postulates which arise in a natural way from the properties of the ratio in which a point divides a segment in elementary geometry. It is shown that the set can be so metrized that f is the quotient of two distances. Limiting processes, orientation, and f-preserving transformations are studied, and other sets of postulates are considered.

Frederick P. Jenks proposes a set of postulates for Bolyai-Lobachevsky geometry based on the operations of joining and intersecting, and shows that these postulates are sufficient for the usual discussion of betweenness.

Kopf's trisection of the angle is improved by B. J. Topel in such a way that the maximum error is reduced from 15″ to 0.5″.

In the concluding paper, Professor Menger advances a number of interesting suggestions on the logic of doubting, commanding, and wishing. In his logic of the doubtful, a proposition must be contained in one of three mutually exclusive classes, according as it is asserted, doubted, or negated. This logic differs from those ordinarily proposed in that the class of a compound proposition is not uniquely determined by the classes of its constituents; two doubtful propositions may be related to each other in a number of ways, according to the classes in which various of their compounds belong. To introduce option into logic, preference is formalized by using a fixed proposition A (for example, "I shall be glad") and regarding p as desirable when p implies A; it is urged that only doubtful propositions are objects of wishes and commands. Perhaps the most suggestive remark made in connection with optative logic —of which several simple cases are treated—is that the relation "M is preferable to (that is, more valuable than) N" furnishes a partial ordering of economic goods in which ordinal incomparability is transitive.

Especially in the first and third papers, there are numerous misprints; except occasionally in the third, none were noticed which seemed likely to confuse an attentive reader.

F. A. FICKEN

Ebene Kinematik. By W. Blaschke. (Hamburger mathematische Einzelschriften, no. 25.) Leipzig, Teubner, 1938. 56 pp.

"Plane Kinematics" is a subject usually regarded as a part of mechanics. It starts with the simple concept of the plane displacement of a rigid body, and leads on through consideration of velocity and acceleration into technical complications of importance to engineers. In the book under review no such treatment is attempted, although the title might lead one to expect it. On the contrary, dependence on time is not included, so that we have to consider only the simplest kinematical concept, namely, the displacement of a rigid body in a plane. The superstructure based on that concept is essentially not that of the engineer (a scant three pages are devoted to linkages) but of the modern geometer in search of a manifold.

Any displacement of a rigid body in a plane may be specified by three parameters (for example, the components of displacement of any point in the body and the angle of rotation). Hence these displacements constitute a manifold or space of three dimensions. It is with the geometry of this three-space that the book is concerned. We must indicate in some detail how the author realizes this three-space.

Let E be the plane in which the displacements take place, and let S be an euclidean three-space containing E. First consider displacements in E other than pure translations. Any such displacement is a rotation about some point A of E through some angle 2ω. Let us take the point B in S on the normal to E at A, at a distance $\cot \omega$ below E. Then B represents the displacement in the sense that B is uniquely defined by, and uniquely defines, the displacement. Actually, the author gives a very simple geometrical construction by which the displacement may be found when B is given, but we shall not describe it here.

Thus the totality of displacements in E (omitting translations) corresponds to the whole of the euclidean space S. Translations correspond to points at infinity of S (with the exception of points at infinity in E), and so the complete representative space for all displacements is a projective three-space P, from which, however, the line at infinity in E is omitted.

For the analytic discussion of the space P, rectangular Cartesian coordinates x, y, z are used (E is $z=0$) and homogeneous coordinates p_0, p_1, p_2, p_3, where $x=p_2/p_1$, $y=p_3/p_1$, $z=p_0/p_1$. The exclusion of the line at infinity in E from P permits the normalizing condition $p_0^2+p_1^2=1$, so that we may write $p_0=\cos\lambda$, $p_1=\sin\lambda$; then λ, p_2, p_3 define a point in P or, equivalently, a displacement in E.

An algebra of displacements is obtained by means of biquaternions of the form $p=p_0e_0+p_1e_1+\epsilon(p_2e_2+p_3e_3)$, where p_0, p_1, p_2, p_3 are real or complex numbers (the homogeneous coordinates as above) and $e_0, e_1, e_2, e_3, \epsilon$ satisfy $e_0=1$, $e_1^2=e_2^2=e_3^2=-1$, $e_2e_3=-e_3e_2=e_1, \cdots, e e_1=e_1e, \cdots, \epsilon^2=0$. It is evident that the product of two biquaternions of the above form is a biquaternion of the same form. The conjugate $\bar{p}$ of p is obtained by changing the signs of the coefficients of e_1, e_2, e_3.

Now if a displacement p is applied to a rigid body, a point initially at l is displaced to r, where $r=\bar{p}lp$. If this displacement is followed by a second displacement q, then the final position of the point is t, where $t=\bar{p}^*lp^*$, in which $p^*=pq$; in fact, the resultant of two displacements is their product.

The "geometry" of the space P consists of those properties of figures in P which remain invariant under certain transformations. These transformations are as follows. Let us apply in order a displacement b_l, a displacement p, and a displacement b_r. The resultant is a displacement $p^*=b_lpb_r$. This is the required transformation of P into itself: these transformations form a group G_6, since each of b_l, b_r depends on three parameters. These transformations leave invariant the points $(0, 0, i, 1)$, $(0, 0, 1, i)$ and the planes $x_0 \pm ix_1=0$, where x_0, x_1, x_2, x_3 are homogeneous coordinates. The resultant geometry the author calls *quasielliptic*, on account of a limiting connection with elliptic geometry.

The preceding remarks give some idea of the first half of the book (Algebraischer Teil). The second half of the book deals with the differential geometry of the quasielliptic space P. A curve is given by writing the homogeneous coordinates p_0, p_1, p_2, p_3 as functions of a parameter λ. The privileged basic parameter is that λ which appears in connection with the normalizing condition $p_0^2+p_1^2=1$, so that the equations of the curve are $p_0=\cos\lambda$, $p_1=\sin\lambda$, $p_2=p_2(\lambda)$, $p_3=p_3(\lambda)$. Quasicurvature and quasitorsion are defined. Surfaces are also considered and curves on surfaces.

The book is based on lectures delivered in 1938 at the University of Hamburg. One may smile at the encouraging prefatorial remark: "An Vorkenntnissen ist nur ein wenig analytische Geometrie und Infinitesimalrechnung erwünscht." For the book is not easy reading, seeming to lack purpose, so that the reader may well ask himself from time to time: "What is the goal of all this?" However, there is much meat in the little volume, as we might expect from the reputation of its author in geometry.

In addition to the regular exposition, there are a number of sections headed "Aufgaben und Lehrsätze."

J. L. SYNGE

Probability, Statistics, and Truth. By Richard von Mises. Translated by J. Neyman, D. Scholl, and E. Rabinowitsch. New York, Macmillan, 1939. 16+323 pp.

This translation follows closely the original, *Wahrscheinlichkeit, Statistik und Wahrheit*, 2d edition, 1936, which I reviewed in the Journal of the American Statistical Association (vol. 31 (1936), pp. 758–759). However, in the preface the author states, "I have added several paragraphs in the English edition (pp. 141–147). These deal with certain investigations of A. H. Copeland, E. Tornier, and A. Wald, which were published after the appearance of the second German edition."

The purpose of Mises is to present in language as non-technical as possible the

fundamental principles of probability with applications to statistics and mechanics. In my opinion, this has been done with marked success. A reader with some background in science, philosophy, or logic should be able to read with understanding the greater part of the book, even though his available mathematics is extremely limited.

Instead of basing a theory of probability upon "equally likely cases," Mises has proposed the use of "*collectives*," sequences of numbers, such as 4, 6, 1, $\cdots$, which might be obtained from throwing a die—these sequences required to satisfy two requirements: Axiom I. If, among the first n numbers, a specified number appears m times, then as n increases indefinitely m/n approaches a limit. Axiom II. "*Randomness*." If from such a sequence, a subsequence is formed by any "place selection" which picks out a term without knowledge of its value, then the foregoing limits are unchanged. This might be looked upon as an axiomatic approach in which Mises *considers only those sequences of numbers or vectors which conform to the above two axioms*. In the "Third Lecture," Mises discusses in detail objections that have been urged against his theory, some authors declaring collectives non-existent. Mises states (p. 143): "Copeland and, later, Wald have proved the correctness of the following proposition for which Copeland found the proof for the case of two attributes only, and which Wald generalized: Given an arbitrary enumerable set of place selections, it is possible to define a collective (even a non-enumerable set of such collectives), in which the relative frequencies of particular attributes tend to the limits prescribed by the given distribution and this is not affected by any of the place selections in the given set. I have interpreted my randomness axiom by saying that the limiting values of the relative frequencies should remain insensitive to all place selections occurring in the particular problem under consideration. It is clear that in any special problem the number of place selections is either finite or, at the most, enumerable. It is even possible to postulate that the system of *all* definable selections is at most enumerable. This postulate agrees well with the modern formal construction of logic. . . . A certain restriction of these propositions is necessary only if the set of attributes considered is *infinite*, ... On the whole, the very thorough and ingenious investigations of Copeland and Wald can be said to dispel completely all the mathematical objections against the use of the randomness axiom which we have previously mentioned." Among the 103 numbered references in the appendix—some referring to more than one author—are several which discuss the foundations of the Mises theory. Another very recent publication is a set of papers by P. Cantelli, W. Feller, M. Fréchet, R. de Mises, J. F. Steffensen, and A. Wald on *Les Fondements du Calcul des Probabilités*, Actualités Scientifiques et Industrielles, no. 735, papers written for the Geneva Conference on the Théorie des Probabilités, 1937.

Mises distinguishes his theory from that based upon "equally likely cases," from theories involving Axiom I, without incorporating Axiom II, and from theories admitting Axiom II in very restricted form. He also takes the position that *probability is not a branch of the theory of additive set functions*. "It remains in all circumstances a theory of certain observable phenomena, which are idealized in the concept of a collective." Mises finds a distinction here that some others have failed to find. At all events, his *collective must first be set up*. After that is done, other branches of mathematics may be used to solve a proposed problem. According to Mises, *no probability exists* unless it is possible to set up a *collective* to describe it.

This review, written for the Bulletin has stressed theoretical considerations. But in Mises' book, the gentle reader will find a delightful non-technical introduction to probability, with carefully selected simple illustrations and a minimum of formulas and mathematical display. He will be led to the heart of many of the most interesting

and important problems in probability itself, and then on to statistics in general and to the statistical problems of theoretical physics or mechanics. Some section headings in the latter topics are the following: Linked events, Lexis theory and the laws of large numbers, Mendel's theory of heredity, Industrial statistics, Galton's board, The second law of thermodynamics, Machines dependent on chance, Small causes and large consequences, Kinetic theory of gases, Brownian motion, Entropy theory and Markoff chains, Radio-active radiations, Quantum theory, The renunciation of causality, Heisenberg's uncertainty principle.

EDWARD L. DODD

La Notion de Point Irrégulier dans le Problème de Dirichlet. By Florin Vasilesco. (Actualités Scientifiques et Industrielles, no. 660.) Paris, Hermann, 1938. 61 pp.

This is the twelfth of the booklets containing "Exposés sur la Théorie des Fonctions" published under the direction of Paul Montel as a part of the series named above. This number contains an interesting compilation of the results of some recent researches on the Dirichlet problem. The discussion makes frequent use of material treated in another booklet by the same author: *La Notion de Capacité* (Actualités Scientifiques et Industrielles, no. 571, 1937).

The first chapter is devoted to a brief discussion of artificial (that is, removable) singularities of harmonic functions of three variables. The author takes as his starting point a theorem to the effect that a function continuous at a point P and bounded and harmonic elsewhere in the neighborhood of P is harmonic at P. As this theorem is attributed to Picard (1923) the author is evidently unfamiliar with an earlier paper by Bôcher in which the same result is established (this Bulletin, vol. 9 (1903), p. 455 ff.; the priority of Bôcher's theorem has already been noted by Raynor, ibid., vol. 32 (1926), p. 537 ff. and by Kellogg, ibid., vol. 32 (1926), p. 664 ff.).

The second chapter, which is the longest of the pamphlet, is devoted to the study of the solution of the generalized Dirichlet problem. The author discusses such topics as conditions for regularity and irregularity of boundary points (especially those expressed in terms of capacity or the conductor potential) barriers, the generalized Green's function, Lebesgue's example of an irregular point, and Kellogg's lemma and some of its corollaries.

Chapter III is devoted mainly to a discussion of the results concerning balayage published by Frostman in 1935.

Chapter IV contains a brief summary of the contents of a booklet by de la Vallée Poussin, *Les Nouvelles Méthodes de la Théorie du Potentiel et le Problème de Dirichlet Généralisé* (Actualités Scientifiques et Industrielles, no. 578, Paris, 1937). One of the topics discussed is the lightening of the requirement of continuity at a multiple boundary point of a spatial domain. Apparently both de la Vallée Poussin and Vasilesco have overlooked an earlier discussion of the Dirichlet problem for three dimensional domains with multiple boundary points (Perkins, Transactions of this Society, vol. 38 (1935), p. 106 ff.).

The remaining chapters of the booklet contain brief sketches of some of the most recent work on topics connected with the Dirichlet problem. Much of this exposition is based on researches published in 1938. Chapter V is concerned principally with Marcel Riesz's notion of generalized potentials of order α (an extension of earlier work by Frostman); Chapter VI is devoted primarily to an account of recent work on balayage by Brelot. Chapter VII is unique in that it makes use of logarithmic capacity; it contains some results (due mainly to Frostman) concerning functions of a complex variable.

An extensive bibliography is given at the end of the booklet.

A few typographical errors were noted. The footnote on page 19 is clearly intended to refer to the first paragraph after the italicized statement of the theorem concerning continuity. In the second formula on page 31, for γ read λ. In the first theorem of section 28, page 46, for (*d'ordre* α, $0<\alpha\leqq 2$) read (*d'ordre* α, $0<\alpha\leqq 2$).

The author assumes that the reader has some knowledge of modern potential theory, especially the theory of capacity. The reviewer feels that even though the pamphlet may not be intended for novices, the discussion of Kellogg's lemma (pages 20–23) could be made clearer by a few prefatory remarks recalling the distinction between Wiener's and de la Vallée Poussin's definitions of capacity.

Despite some imperfections of detail, the pamphlet is unquestionably a notable contribution to the literature on modern potential theory, and will undoubtedly prove highly valuable to many students of this subject.

FRED W. PERKINS

Les Fonctions de Matrices. I. *Les Fonctions Univalentes.* By Hans Schwerdtfeger. (Actualités Scientifiques et Industrielles, no. 649.) Paris, Hermann, 1938. 58 pp.

This monograph should be of particular interest to the student of analysis; the introductory algebraic treatment, in fact, is suggested and delimited by problems in the field of differential equations.

The opening chapter is an exposition of those properties of matrices that are standard equipment for every geometer and algebraist. It is in Chapters II and III that one finds the author's own contributions.

Chapter II deals mainly with the following problem: Given a set of single-valued functions $f(\xi)$, $g(\xi)$, $\cdots$ of the real or complex variable ξ, and an n-rowed square matrix A with fixed coordinates in the complex field, it is required to define n-rowed square matrices $f(A)$, $g(A)$, $\cdots$ so that the following requirements are satisfied:

1. If $f(\xi)$ is a constant α for every ξ in the set S, then $f(A)=\alpha E$ (E the identity matrix) for every A whose characteristic roots lie in the set S.

2. To the functions $f(\xi)\pm g(\xi)$ and $f(\xi)g(\xi)$ correspond respectively the matrices $f(A)\pm g(A)$ and $f(A)g(A)$.

3. To the function $g(f(\xi))$ corresponds the matrix $g(f(A))$, assuming that the variation of ξ is so restricted that the values η of $f(\xi)$ lie in the domain of definition of $g(\eta)$.

When each function is a polynomial $\sum\alpha_i\xi^i$, the problem is solved by choosing for $f(A)$ the polynomial $\sum\alpha_iA^i$. For a large class of functions the author shows that the definition of $f(A)$ can be obtained from the "polynomial definition" by using an interpolation formula of Hermite. In particular, if $f(\xi)$ is a single-valued function such that $f(\lambda_i)$, $f'(\lambda_i)$, $\cdots$, $f^{(n-1)}(\lambda_i)$ exist, where λ_i is a root, of multiplicity n_i, of the characteristic equation of A, then the author obtains for $f(A)$ the formula $f(A)=\sum_{i=1}^{m}A_i\sum_{r=0}^{n_i-1}(1/r!)f^{(r)}(\lambda_i)(A-\lambda_iE)^r$, where A_i and m are respectively the Frobenius covariants and the number of distinct characteristic roots of the matrix A. In the final section of Chapter II the author shows that from this formula one obtains a definition for $f(At)$, t a real or complex variable, if one replaces $f^{(r)}(\lambda_i)$ by $f^{(r)}(\lambda_it)$. For $f(\xi)=e^{\xi}$, in particular, the author's formula leads to $e^{At}=\sum_{i=1}^{m}e^{\lambda_it}A_i\sum_{r=1}^{n_i-1}(t^r/r!)(A-\lambda_iE)^r$.

The function e^{At} occupies a central position in Chapter III. Here the author applies the results of Chapter II to the theory of systems of differential equations. As one would expect, the matrix notation enables him to exhibit solutions in a compact and elegant form.

CHARLES HOPKINS

Die Pellsche Gleichung. By Werner Weber. (Deutsche Mathematik, supplement no. 1.) Edited by Theodor Vahlen. Leipzig, Hirzel, 1939. 8+151 pp.

In his preface the author calls this an exhaustive text devoted to the mathematical properties of the Pell equation $t^2-Du^2=h$, $h=\pm 1, \pm 4$, that can serve on the one hand as an introduction, but which is intended primarily as a reference work "in dem man endlich einmal alles findet, was anderswo mit schöner Selbstverständlichkeit als bekannt vorausgesetzt wird." He seeks, by deriving the properties of the equation from those of the units in the orders of a quadratic number-field, as it were to clear a path through the forest that has arisen in the literature because of the variety of hypotheses that different writers have imposed on D or on h. But his exposition offers little better than a list of the trees.

The first of four parts is entitled "Pell Equation and Units." It is difficult to see for what class of reader this part is written. A glance at its 146 theorems might lead one to think that it was addressed to utter novices, for all but a few of the proofs are trivial deductions from preceding material. But the triviality is due partly to presupposing at the outset familiarity with the elements of the theory of quadratic fields, with the Dirichlet theory of units, with orders, and at one place with rudiments of ideal theory, topics hardly within the ken of an utter novice, and partly to a too rigid adherence to the Satz-Beweis form of presentation. When we reach the third part of the work we find a helpful résumé, which however would have been much more useful had its essence appeared earlier as an objective and motivation for the first part.

The second part, "Pell Equation and Binary Quadratic Forms," is much better reading, though at times repetitious. Here we find an accurate, fully detailed account of the continued fraction expansion of the roots of reduced indefinite binary quadratic forms, Kronecker forms being treated first and later specialized successively to Gauss forms, to forms with last coefficient -1, and to Dx^2-y^2. The connection of the Pell equation with automorphs of the forms is clearly brought out. Values of h other than ± 1, ± 4 with $|h|<2D^{1/2}$ are treated as they arise naturally in connection with continued fractions.

In the fourth part, under the title "Practical Aids," is a list of published tables connected with the Pell equation and a study of their uses. The arrangement is again unfortunately such as seriously to impair the usefulness of the material. 58 tables are listed under 11 categories on the basis of content. Two of the categories contain precisely the same 8 tables, and a third category merely relists 3 of these 8. Several pages are then devoted to an exhaustive presentation of 78 relations (Verkappungen) between the categories. For example it is carefully noted that tables giving the fundamental solution of $t^2-Du^2=+1$ can be used to find the fundamental solutions of $t^2-Du^2=\pm 1$ and conversely. The portion of this material that is of value could easily be put in much briefer and more accessible form.

On the second page of the preface we are told that elementary algorithms for finding the solutions are not included, neither are "more advanced and special" facts, solutions derived from cyclotomy for example. For history, reference is made to four standard sources, the last of which bears the date 1920. The list of tables includes recent publications, but no other mention is made of the (not very extensive) literature since 1920.

A succinct account of some of the material in this work would fill a need. But the expository form here adopted, with its over-emphasis on trivialities, vitiates the author's care in detail, his accuracy of statement, and the volume's typographical excellence.

B. P. GILL

New First Course in the Theory of Equations. By L. E. Dickson. New York, Wiley, 1939. 10+185 pp.

Of the author's three books on the Theory of Equations this new book is most closely akin to his "First Course in the Theory of Equations," the most recent of the three. It is, however, much more than a mere revision of the latter text.

The order of the topics treated has been changed so that now the progress of the reader is from the simpler to the more complex. The number of the problems has been increased and they have been carefully worked out to illustrate the theory with a minimum of computation.

The New First Course, though only slightly longer than the old First Course, lacks little that the latter contained. When such omissions occur they are usually replaced by something more interesting. For instance Waring's formula for the sum of kth powers of the roots of an equation in terms of its coefficients together with its involved proof has been deleted and in its place appears Brioschi's elegant determinant form for the sum for $k=2, 3, 4$.

With the space saved by simplifying the proofs numerous new features are added. There is, for one thing, a more careful approach to determinants and the proofs for the general theory of linear equations are considerably clarified by the introduction of matrix concepts.

The discussion of trisection of angles is made so clear that the following theorem could safely be stated: anyone who seeks a method for trisecting all angles with ruler and compasses alone has not read this book.

There is one new simplification in the treatment of Sturm's functions which considerably shortens numerical computation.

Even if the reviewer were not predjudiced in favor of the author he could not fail to rejoice at the advent of this New First Course.

B. W. JONES

Projektive Geometrie. By Heinz Prüfer. Leipzig, Akademische Verlagsgesellschaft, 1939. 8+314 pp.

The first half of this book is devoted to the usual topics of synthetic geometry of the first and second order in one, two and three dimensions, including polarity. It is based on the operations projection and section, but independently of intuition. Each step is rigorously defined and explained in terms of the axioms used, including continuity. But that space is three dimensional is tacitly assumed without an axiom of closure. The Playfair statement is the form adopted for the parallel axiom. An unusually large amount of material is satisfactorily discussed in these 150 pages. No exercises are provided for the student. No use is ever made of imaginary elements.

Then follows a chapter on metrical geometry, mostly confined to two dimensions. This is particularly well done. The concept of perpendicularity is introduced by axioms; the involution of pairs of perpendicular lines of a pencil and a polarity having no curve of incident elements are the only new ideas needed. These are applied to prove a number of metrical theorems of plane geometry, connected with triangles. For the sake of logical completeness, now follows a chapter on non-euclidean geometry. This is much harder reading; it is logically consistent, but pedagogically is less successful.

Throughout the book figures are used freely, but only as suggestions, never as an essential part of the proof. A chapter on descriptive geometry is hardly more than a sketch; it discusses so many principles in the short space available that a reader would be helpless in trying to apply them to any other than the simplest problems.

The final chapter is on coordinates in one and two dimensions. It is based on the fundamental theorem that three pairs of corresponding elements fix a one-dimensional projectivity, so that the correspondent of any fourth element is uniquely determined. The idea of cross-ratio is not explicitly introduced. After defining addition and multiplication geometrically, it is shown that the rules of ordinary algebra apply. In passing from homogeneous to nonhomogeneous coordinates, the statements are frequently too inclusive; as given they include division by a vanishing coefficient. After showing that loci represented by linear equations in point coordinates are straight lines, the analytic formulation of projectivity is discussed, and also correlation. The determination of the fixed elements is not taken up except in a few particular cases.

The style is on the whole pleasing; the book is easy to read. The printing is excellent, only two typographical errors having been found. The work is provided with a full index.

VIRGIL SNYDER

Contributions to the Mechanics of Solids. (Dedicated to Stephen Timoshenko by his friends on the occasion of his sixtieth birthday anniversary.) New York, Macmillan, 1938. 277 pp.

The preface states that the idea of producing this book to commemorate the sixtieth birthday of Stephen Timoshenko, formerly professor of engineering mechanics at the University of Michigan, was conceived almost simultaneously by several of his present and past associates in colleges and industries. The book consists of a short biography of Professor Timoshenko followed by twenty-eight independent articles on various aspects of "stress and strain," written by prominent men in science and industry on both sides of the Atlantic.

Many of the articles discuss rather particularized engineering phases rather than general principles or analytical mathematics. The average technical reader will find some interesting features in such articles as "Developments in Photoelasticity" (non-mathematical), "Effect of a Flexible First Story in a Building Located on Vibrating Ground," "Dynamic Stability of Railway Trucks," "Use of Orthogonal Functions in Structural Problems," and "Hamilton's Principle and the Principle of Least Action in the Solution of Creep Problems." The above titles illustrate the range and mutual independence of the articles which constitute the book. The reviewer feels that it would have been of more interest and value to have assembled more correlated discussions of topics in the field of the mechanics of solids.

J. K. L. MACDONALD

Foundations of Logic and Mathematics. By Rudolf Carnap. (International Encyclopedia of Unified Science, vol. 1, no. 3.) Chicago, University Press, 1939. 8+71 pp.

This monograph presents in condensed form and with a minimum of formal detail the author's views concerning the relation of logistically formalized calculi to language in the ordinary sense, and concerning the application of such calculi in empirical science. It is a noteworthy contribution to philosophy of science and in particular to analysis of the relationship between pure and applied mathematics, the questions involved being made much more precise and intelligible than would otherwise be possible, through use of the methods of modern symbolic logic.

In many respects the author's views are here modified or clarified in such a way as to remove serious objections previously urged against them.

The "principle of tolerance" is explicitly restricted to uninterpreted logistic calculi and it is said that "a system of logic is not a matter of choice, but either right or wrong, if an interpretation of the logical signs is given in advance." The quoted sentence—by failing to take account of the fact that an interpretation, in advance of *some* formalization, must have a considerable element of vagueness—may even admit too much to the anti-conventionalists. The author adds, however: "It is important to be aware of the conventional components in the construction of a language system."

The purely syntactical method of the author's previous publications is here supplemented by an account of semantics. Designata are admitted not only for concrete terms but also, in some cases at least, for abstract symbols and expressions. Thus predicates are said to designate properties of things (p. 9), (declarative) sentences are allowed to designate "states of affairs" (p. 11), and "functors" are said to be signs for functions (p. 57). (The more usual terminology is "proposition" instead of "state of affairs" and "function symbol" instead of "functor.") The reviewer would prefer a still more liberal admission of abstract designata, not on any realistic ground, but on the basis that this is the most intelligible and useful way of arranging the matter—it would apparently be meaningless to ask whether abstract terms *really* have designata, but it is rather a matter of taste or convenience whether abstract designata shall be postulated.

The point brought out in §16, that a postulate set in the usual mathematical sense must be regarded as added to an underlying system of logic—which, for exactness, must be logistically formalized—is, of course, not new. But it deserves attention, because neglect of just this point has resulted in much misunderstanding concerning the significance of a set of postulates for a particular mathematical discipline.

On page 23, instead of distinguishing between finite and transfinite rules, it would seem to be better to distinguish between effective and non-effective rules. The matter is complicated by the fact that "finite" is often used in this connection substantially as a synonym of "effective." But a rule might well be non-effective without being transfinite in Carnap's sense.

In §14 there appears to be an oversimplification of the relation between logic and arithmetic, partly through failure to make explicit mention of the axiom of infinity, and partly through an unsound use of recursive definition. An example of the latter is Definition 14, which is in effect a schema providing separate definitions for $m+0$, $m+1$, $m+2$, $\cdots$. That this is no definition of the function $+$ may be seen by considering that the sentence, "For all natural numbers x and y, $x+y=y+x$," for example, remains undefined. This section (like most of the monograph) undertakes only to provide an outline statement with omission of formal detail; nevertheless it seems to the reviewer that an unfortunately misleading impression is given.

ALONZO CHURCH

Elements of the Topology of Plane Sets of Points. By M. H. A. Newman. Cambridge, University Press, 1939. 216 pp.

"This book," according to the scholarly description which appears on the jacket, "has the double purpose of providing an introduction to the methods of topology and of making accessible to analysts the simple modern technique for proving the theorems on sets of points required in the theory of functions of a complex variable [separation theorems, for example]."

There is no doubt that for non-topologists at least, many of the proofs of the Jordan separation theorem which have appeared in the literature are either dull or

else too rapid for comfort. In the present work, the Jordan theorem and related topics are treated in such a manner that there can be no complaint on either score. The author first introduces a very sensible treatment of complexes, the "simple modern technique" referred to above. Let us imagine a grating formed in the plane by a number of vertical and horizontal lines. Systems of one- and two-dimensional complexes could hardly be simpler than those consisting respectively of the polygonal lines and polygonal regions formed from the grating; methods of combining complexes could hardly be more geometric and intuitive than those defined in terms of modulo two algebra. Moreover, a given grating can be indefinitely "refined" simply by interpolating more lines. The complexes formed from a grating and its refinements constitute a most useful system: it furnishes two-complexes which will approximate any open set with arbitrary closeness and one-complexes which will separate two disjoint closed sets however near together they are.

An exposition of these simple ideas leads at once to the following lemma, due to Alexander. Let Z be the plane closed by the addition of a single ideal point at infinity. Let x and y be points and F_1, F_2 closed sets in Z. On some grating let κ_1 and κ_2 be one-complexes joining x and y and such that κ_i fails to meet F_i, $(i=1, 2)$. Then if x and y are separated by the set F_1+F_2, the polygon $\kappa_1+\kappa_2$ cannot be the boundary of a two-complex in $Z-F_1F_2$. This lemma, the proof of which requires but a few lines, is shown by the author to be one of the sharpest tools in the theory of separation, if skillfully handled. It is used here to settle quickly the decisive points in such questions as the proof of the Jordan theorem, accessibility, the invariance of dimensionality and regionality, the mapping of a simply connected domain and its "boundary elements" onto a closed circular region.

These topics definitely concern sets in the plane, and form the content of Part II. Part I however contains a discussion of sets in general, metric spaces in particular. An account of the fundamental concepts of topology terminates with the definitions of local connectedness and the intrinsic characterizations of simple closed curves and simple arcs. The characterization of the simple closed curves later forms the basis for easy proofs of the so-called converses of the Jordan theorem.

The last chapter contains an account of connectivity theory "in miniature." Only the first two connectivity numbers enter the discussion: p_0 defined for arbitrary plane sets, and p_1 defined for open plane sets. At first sight it would seem that a theory about p_0 and p_1 could hardly be more than trivial. But this is not the case. There is for example a duality theorem of the Alexander type for closed sets in Z—that is, a relation between $p_0(E)$ and $p_1(Z-E)$ where E is a closed set in Z. Moreover, a proof of the invariance of p_1 is not entirely trivial. We believe that here it would have been well to state precisely the meaning of invariance in Theorem 4.1. This theorem asserts bluntly that p_1 is a topological invariant. But one gathers from Part I that a topological invariant of a metric space E is a property that is possessed by every metric space homeomorphic to E. The theorem in question, however, can only mean that for two open plane sets which are homeomorphic, the values of p_1 must be equal. It is possible that the alert student will at this point sense the limitations of the connectivity theory that is presented, and will begin to wonder about suitable definitions for general spaces. Thus he will enter a stream of thought which has passed through fertile regions but which seems as yet not to have reached its ultimate destination.

It seems to the reviewer that the author has accomplished admirably his twofold purpose. The student will find here an introduction to combinatorial topology which is simple, yet thoroughly scientific. To be sure he will not encounter the delights of the torus and the Möbius strip, but he will see the theory of complexes usefully employed. The analyst will find the necessary separation and mapping theorems

treated with skill and clarity. It should be added that he will also find a short readable proof of the Cauchy formula $\int_J f(z) = 0$ in which $f(z)$ is assumed to be regular over the inner domain D of the rectifiable simple closed curve J and continuous in $D+J$. Here again the power of Alexander's lemma shows itself in solving rapidly the separation problems that arise.

P. A. SMITH

Étude Critique de la Notion de Collectif. By Jean Ville. (Monographies des Probabilités, no. 3.) Paris, Gauthier-Villars, 1939. 144 pp.

Professor Ville has written an interesting and valuable discussion of the concept of a collective, upon which many mathematicians found the theory of probability. The author discusses systems of play in detail, and generalizes this idea to that of a "martingale." This leads to a new criterion for the exclusion of sequences from probability discussions, that is, to a new definition of collective. Any given set of sequences of probability 0 can be excluded by this new criterion, whereas the system criterion, used by Copeland, Popper, Reichenbach, Tornier, Wald, can be used only to exclude certain sets of sequences (necessarily of probability 0). Ville extends the definition of a martingale to the case of a stochastic process depending on a continuous parameter, and shows that some of his sequence results go over.

It is unfortunate that this book, which contains much material which clarifies the subject, should contain so much careless writing. This ranges from uniformly incorrect page references to mathematical errors. Thus (p. 46) it is claimed (and used in a proof) that every denumerable set is a G_δ. The author's main theorem on systems is not as strong as earlier results with which he is apparently unfamiliar. (Cf. Z. W. Birnbaum, J. Schreier, Studia Mathematica, vol. 4 (1933), pp. 85–89; J. L. Doob, Annals of Mathematics, (2), vol. 37 (1936), pp. 363–367.) His discussion of random functions is inadequate and obscure, for example, his demonstration that his main theorem on martingales does not go over to the continuous process uses as an example a measure on function space not in accordance with the usual definition of probability measures on this space.

A specialist who can overlook such slips will find many stimulating ideas in this book. Other readers can profit by the comparative analysis of the different criteria for collectives, and by the discussion of martingales.

J. L. DOOB

Problems in Mechanics. By G. B. Karelitz, J. Ormondroyd, and J. M. Garrelts. New York, Macmillan, 1939. 9+271 p.

This is a collection of nearly 800 problems in statics, kinematics and dynamics. Some two thirds are based on those compiled by the late I. V. Mestchersky, of the Polytechnic Institute of St. Petersburg. The authors have not only translated these, but have replaced the metric by English engineering units and given them a background suitable to American students.

The book is intended to supplement a first course in mechanics as applied to engineering. Thus the problems vary from simple exercises in resolution of forces and falling bodies to those on tensions in cables and curvilinear motion under central forces. Many will provide hard practice in the application of mechanical principles, but none are of the puzzle type. Only rudimentary calculus or differential equations and no knowledge of Lagrange's equations is assumed. As is the case in actual engineering practice, with few exceptions the problems are reducible to those in one or

two dimensions. The setting in most cases will appeal to the student as interesting and useful.

A fifty page résumé of the principles of mechanics is added. This facilitates references in the solutions which are given to some typical problems. Answers are given in practically all cases.

The book may be recommended to teachers who wish more problems than those found in the usual textbook in elementary applied mechanics.

PHILIP FRANKLIN

Elementary Matrices and some Applications to Dynamics and Differential Equations. By R. A. Frazer, W. J. Duncan, and A. R. Collar. Cambridge, University Press, 1938. 16+416 pp.

The authors are primarily interested in developing the theory of matrices as a tool, especially to be applied to aerodynamics. Consequently it is only the first four chapters which are concerned with matrices as such, while the remaining nine chapters are devoted to applications. Numerous examples occur throughout the work and much stress is put on the technique of numerical computation. An elementary knowledge of determinants is assumed at the outset.

In the first three chapters are found very satisfactory treatments of most of the standard theorems on matrices, including such topics as polynomials and power series of matrices, the Cayley-Hamilton theorem, Sylvester's theorem, and canonical forms for matrices. No treatment is given, however, of the reduction of a matrix to canonical form under collineatory transformations, although the fundamental results are stated and the reader is referred to a proper place in the literature. The fourth chapter is devoted to miscellaneous numerical methods for finding the reciprocal, high powers, and latent roots of a matrix, together with a few applications.

Chapters five and six treat systems of linear ordinary differential equations with constant coefficients. The fact that matrices can be used in treating systems of this kind is indubitably the reason that matrices are so important to the aeronautical engineer. The treatment given is unsatisfactory to the mathematician, because it is not shown that a complete set of solutions is given by the methods described (although this is indeed the fact); nor is there an adequate reference to fill the gap. Yet this very matter constitutes the most difficult part of the whole theory (cf. p. 168).

Chapter seven describes miscellaneous applications to linear differential equations with variable coefficients.

The rest of the book is primarily concerned with dynamics and becomes more and more technically involved in aeronautical problems as the book progresses to its close, the last chapter being entitled "pitching oscillations of a frictionally constrained aerofoil." It should be remembered, however, that much of this technicality is that of language only. On page 267, for example, we find a section entitled "the equations of motion of an aeroplane." The authors then go on to say that they are deriving the equations of motion for any rigid body, which merely for definiteness is supposed to be a (rigid) aeroplane in flight!

The book will probably prove invaluable to the student of aerodynamics. It should also be a stimulating reference book for the really good undergraduate who is assailed by doubts as to the practical applications of mathematics.

D. C. LEWIS

NOTES

Two new volumes have recently appeared in the series American Mathematical Society Colloquium Publications. Volume 23 is *Orthogonal Polynomials* by Professor Gabor Szegö of Stanford University. This book consists of 10+401 pages and the list price is $6.00. Volume 24 is *Structure of Algebras* by Professor A. A. Albert of the University of Chicago. This volume consists of 12+210 pages and the list price is $4.00. Both of these volumes may be obtained from the New York office of the Society. A discount of twenty-five per cent is allowed to members.

The University of Toronto has decided to sponsor a series of books under the title *Mathematical Expositions*. The first concern of each author will be to present his subject matter in readable form while supplying the natural background and calling attention to significant ideas. Each volume will run to about 125 pages. The first two will deal with the *Foundations of Geometry* and the *Infinite in Mathematics*. It is expected that one of these will appear by the end of the present year. Arrangements have been made for the preparation of certain other volumes. It is hoped that the price will not be more than $2.00 with a reduction of twenty-five per cent to members of the American Mathematical Society.

Applications are being received for Benjamin Peirce Instructorships in Mathematics at Harvard University for the academic year 1940–41. Candidates should ordinarily have the doctorate or its equivalent. Applications should be sent to, and further information may be received from, the Chairman of the Department of Mathematics.

Yale University offers each year a number of Sterling research fellowships, some of them open to mathematicians. For the year 1940–41 the normal stipend will be $1800. Applicants must have received the degree of Doctor of Philosophy or its equivalent from approved universities in the United States or foreign countries. In general, preference is given to applicants of not more than thirty-five years of age, but older persons are eligible. Applications should be received before March 1; blanks may be obtained from the Graduate School office, Yale University.

At a meeting of the American Institute of Electrical Engineers on January 24, 1940, New York City, the Edison and Hoover Medals will be awarded. Following the presentation Professor E. Fermi of Columbia University will give a lecture on *Atomic disintegration*.

Mr. A. E. Ingham of Cambridge University has been granted a Leverhulme fellowship for research in mathematics.

Professor G. N. Watson of the University of Birmingham has received an honorary doctorate from the University of Edinburgh.

Professor G. D. Birkhoff of Harvard University has been elected a foreign honorary member of the Academia de Ciencias Exactas, Fisicas y Naturales de Lima.

Professor Harold Hotelling will give a series of six lectures in November at the Institut Henri Poincaré in Paris, dealing with modern methods of combining observations on a multiplicity of variates and testing hypotheses. He will then spend three

months in India advising government officials and lecturing on statistics in Calcutta, Madras, and other cities, under the auspices of the Indian Statistical Institute.

Professor Vladmir Karapetoff of Cornell University has retired with the title emeritus. During 1939–40 he will give a course of lectures at Stevens Institute of Technology.

Professor Eyraud has been appointed to a professorship at the University of Lyon.

Assistant Professor J. C. Brixey of the University of Oklahoma has been promoted to an associate professorship.

Professor W. A. Cordrey of East Carolina Teachers College has been appointed to a professorship at Southeastern Louisiana College.

Dr. G. M. Ewing of the University of Missouri has been promoted to an assistant professorship.

Dr. B. E. Gatewood of the University of Wisconsin has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Dr. C. H. Graves of Pennsylvania State College has been promoted to an assistant Professorship.

Dr. H. S. Kaltenborn of the University of Texas has been appointed to an assistant professorship at Louisiana Polytechnic Institute.

Assistant Professor Jack Levine of North Carolina State College has been promoted to an associate professorship.

Dr. D. C. Lewis of Cornell University has been appointed to an assistant professorship at the University of New Hampshire.

Dr. W. C. McDaniel has accepted a position at Southern Illinois Normal University.

Associate Professor Paul Muehlmann of Xavier University, Cincinnati, has been promoted to a professorship.

Assistant Professor P. K. Rees of New Mexico College of Agriculture and Mechanic Arts has been appointed to an assistant professorship at Southern Methodist University.

Dr. J. B. Rosser of Cornell University has been promoted to an assistant professorship. He will be on leave of absence during 1939–40 at Princeton University.

Assistant Professor R. C. Stephens of Knox College, Galesburg, Illinois, has been promoted to an associate professorship. During the year 1939–40 he will be on leave.

Assistant Professor E. C. Stopher of Ashland College, Ashland, Ohio, has been promoted to an associate professorship.

Professor H. L. Turrittin of the College of Mines and Metallurgy at El Paso, Texas, has been appointed to an assistant professorship at the University of Minnesota.

Professor T. L. Wade of Mercer University has been appointed to an assistant professorship at the University of Alabama.

Dr. Abraham Wald, formerly of Vienna, has been appointed as lecturer at Columbia University for the academic years 1939–41.

The following members of the American Mathematical Society are in residence at the Institute for Advanced Study for all or part of this academic year: Dr. Warren Ambrose, Dr. Valentin Bargmann, Dr. P. G. Bergmann, Professor Garrett Birkhoff of Harvard University, Professor A. D. Campbell of Syracuse University, Dr. M. M. Day, Dr. Paul Erdös, Professor W. W. Flexner of Cornell University, Dr. R. H. Fox, Dr. Kurt Gödel formerly of the University of Vienna, Dr. P. R. Halmos, Professor S. C. Kleene of the University of Wisconsin, Dr. R. S. Phillips, Dr. Abraham Schwartz, Professor H. L. Smith of Louisiana State University, Professor P. A. Smith of Barnard College, Professor R. C. Stephens of Knox College, Dr. Henry Wallman, and Mr. J. S. De Wet.

The following persons are also to be at the Institute for Advanced Study during a part or all of the current academic year: Dr. W. H. Barkas, Dr. A. T. Brauer formerly of the University of Berlin, Dr. C. H. Dowker, Professor Guido Fubini formerly of the University of Turin, Mr. A. E. Ingham of the University of Cambridge, and Professor Henry Margenau of Yale University.

The following appointments to instructorship are announced: University of California at Los Angeles: Dr. P. G. Hoel; Cornell University: Mr. Theodore Hailperin, Dr. Mark Kac, Dr. G. E. Schweigert; University of Detroit: Mr. W. A. Vezeau; Hofstra College, Hempstead, New York: Dr. J. E. Eaton; University of Michigan: Dr. W. T. Scott; University of Minnesota: Dr. H. A. Arnold; University of Nevada: Dr. E. P. Vance; University of Rochester: Dr. J. W. Green; Tennessee Polytechnic Institute: Mr. J. A. Ward; Texas Technological College: Dr. L. F. Ollmann; Dr. R. K. Wakerling.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

397. Ben Dushnik and E. W. Miller: *Concerning similarity transformations of linearly ordered sets.*

This paper is concerned with the following questions: (1) Is every linearly ordered infinite set A similar to a proper subset of itself? (2) Let A be a linearly ordered set such that if f is any similarity transformation of A into a subset of itself, then $f(a) \geqq a$ for every a in A. Is it true that every such set A is well ordered? It is shown that the answer to each of these questions is in the affirmative if A is denumerable. An example is constructed to show that these conclusions need not hold if A is non-denumerable. (Received August 23, 1939.)

398. J. J. DeCicco: *The circular group in an infinite plane of the Kasner space.*

A horn-set (γ) consists of all the curves of the plane which pass through a given point in a common direction and have the same curvature γ. Let $x=2\gamma'$, $y=5^{1/2}\gamma''$, $z=2(\gamma'''-\gamma^2\gamma')$, where $(\gamma', \gamma'', \gamma''')$ are the first three derivatives of the curvature γ of any curve C of a horn-set (γ) at the common point. A horn-set (γ) may be regarded as a three-dimensional space, called the Kasner space K_3, where any point of K_3 is a curve $C(x, y, z)$ of the horn-set (γ). The group of conformal transformations induces a special affine five-parameter group G_5 between the Kasner spaces. The ∞^2 planes $4z=-b^2x+4by+4c$ are called the infinite planes of K_3. If $u=-x$, $v=bx/2-y$ denotes any point of an infinite plane, the group G_5 induces the metric $M_{12}=(u_2-u_1)^2/(v_2-v_1)^2$ between any two points of an infinite plane. The circles are the semicubical parabolas $(u-a)^2=R(v-b)^2$. The circular group is $U=au+c$, $V=bv+d$. A minimal characterization of this group is also obtained. (Received August 2, 1939.)

399. D. W. Hall (National Research Fellow) and G. T. Whyburn: *An analysis of arc-preserving transformations.*

Arc-preserving transformations have previously been defined and studied by one of the authors (G. T. Whyburn, American Journal of Mathematics, vol. 58 (1936), pp. 306–312) and the present paper results from a continuation of that investigation. Let K denote the set of all cut points and end points of a compact locally connected continuum A. A point of $A-K$ is called an internal point of the cyclic element of A containing it. Then (i) if $T(A)=B$ is continuous, in order that T be arc-preserving the following conditions are necessary and sufficient: (a) for each true cyclic element E_b of B there exists a true cyclic element E_a of A mapping onto E_b topologically under

T and such that T^{-1} is single-valued on the set $T(H)$, where H is the set of all internal points of E_a, and (b) the image of each cyclic chain in A is a cyclic chain in B. (ii) If $T(A)=B$ is arc-preserving, then the image of each true cyclic element E_a of A is a point, a free arc of B lying in no true cyclic element of B, or a true cyclic element E_b of B in which case $T(E_a)=E_b$ is topological. (Received September 2, 1939.)

400. Edward Kasner and J. J. DeCicco: *Characterization of the conformal group by horn angles of second order.*

This is a continuation of the paper by Kasner, *Characterization of the conformal and equilong groups by horn angles* (Duke Mathematical Journal, vol. 4 (1938), pp. 95–106). A horn angle is the configuration formed by two curves of a plane which pass through a common point (the vertex) in a given direction. It is said to be of order n if the curved sides have $n+1$ but not $n+2$ points in common at the vertex. Every horn angle of order n has a unique absolute conformal differential invariant of order $2n+1$. (See Kasner, *Conformal geometry*, Proceedings of the Fifth International Congress of Mathematicians, vol. 2 (1912), p. 81.) This is called the conformal measure of the horn angle. In this paper, it is proved that the conformal group is characterized by the preservation of the conformal measure of every horn angle of *second* order. In the Duke paper, the same thing was done for a horn angle of first order. Finally, it is found that the conformal group is characterized by the preservation of either of the two relative conformal invariants for a horn angle of second order, namely, one of the third and the other of the fifth order. (Received August 2, 1939.)

401. E. R. Lorch: *Spectral analysis of weakly almost periodic transformations in reflexive vector spaces.*

It is proved that in a weakly compact space, a bicontinuous linear transformation V is weakly almost periodic if and only if it is uniformly bounded, $|V^n| \leqq K$, $n=0, \pm 1, \pm 2, \cdots$. If the underlying space $\mathfrak{B}$ is reflexive and if V is weakly almost periodic in $\mathfrak{B}$, then the following spectral analysis of V is possible: For every θ in the interval $-\pi<\theta\leqq\pi$, there exist two closed linear manifolds $\mathfrak{M}_\theta$ and $\mathfrak{N}_\theta$ such that $\mathfrak{M}_\theta$ and $\mathfrak{N}_\theta$ have only the element 0 in common and together span the space $\mathfrak{B}$. These manifolds reduce V. Conditions are given that $\mathfrak{M}_\theta$ and $\mathfrak{N}_\theta$ be disjoint thus generating a projection. The exact nature of the above analysis may best be indicated by applying it to a well known special case: If V is a unitary transformation in a Hilbert space, then $\mathfrak{M}_\theta$ and $\mathfrak{N}_\theta$ yield immediately the resolution of the identity of V. (Received August 29, 1939.)

402. Deane Montgomery and Leo Zippin: *Topological group foundations of "rigid" space geometries.*

The axioms of Hilbert on the foundations of plane geometry (*Grundlagen der Geometrie*, Mathematische Annalen, vol. 56 (1902); also, Appendix IV, 7th edition of *Grundlagen der Geometrie* (Teubner)) cannot be carried over, without some addition, to a characterization of 3-space geometries. The present paper deals with ordinary 3-space and a system G of homeomorphisms satisfying Hilbert's axioms I and III and the following II*: there exists at least one point p such that the group G_p, leaving p fixed, is a proper subgroup of G and furthermore such that the orbits $G_p(x)$, for at least one sequence of points converging to p, are two-dimensional. This asks less than Hilbert's II in that it is postulated of one point of space; it asks incomparably more at that point. It is shown that of any three systems satisfying these conditions at

least two are abstractly identical. There are, of course, two cases, euclidean and hyperbolic. (Received August 2, 1939.)

403. Gabor Szegö: *On the gradient of solid harmonic polynomials.*

Let $u(x, y, z)$ be a real harmonic polynomial of the nth degree ($n \geqq 4$) which satisfies the inequality $|u(x, y, z)| \leqq 1$ in the unit sphere $x^2+y^2+z^2 \leqq 1$. Then $|\text{grad } u| \leqq 2n\{1/1-1/3+\cdots+(-1)^n/(2n-1)\}+[(-1)^n-1]/2$ holds for $x^2+y^2+z^2 \leqq 1$, and this is the precise upper bound. This is the analogue of a previous theorem of the author (Königsberger Gelehrte Gesellschaft, 1928, pp. 59–70) on the gradient of harmonic polynomials in two variables, which in turn is a more precise form of S. Bernstein's classical theorem on trigonometric polynomials. (Received August 15, 1939.)

404. V. W. Adkisson and Saunders MacLane: *On extending maps of plane Peano continua.*

Let the Peano continuum M be situated on a sphere S and let the homeomorphism T map M into a subset M' of a sphere S'. Then T can be extended to a homeomorphism which carries the whole sphere S into S' if and only if any two triods of M which have the same sense on S are mapped by T into triods of M' having the same sense on S'. This theorem is established by using Kline's analysis of sense on closed curves and by other arguments employed in Gehman's condition for the extendability of T (Transactions of this Society, vol. 28 (1926), pp. 252–265). In the condition for extendability, essentially only those triods with vertices at cut or split points of M need be considered. The proof of this fact requires the construction of a maximal triply connected subcontinuum of M containing any specified θ-graph in M, as well as certain existence theorems for triods. (Received September 19, 1939.)

405. R. P. Agnew: *On continuity and periodicity of measurable functions.*

Let $x(t)$ denote a complex-valued function measurable over $-\infty < t < \infty$. Then there is a decreasing sequence $H_1 > H_2 > \cdots$ converging to zero with the following property. If $h_1, h_2, \cdots$ is a real sequence with $|h_n| < H_n$ for each $n = 1, 2, \cdots$, then for each t_0 with the exception of a null set (which may depend upon the particular sequence h_n), $x(t)$ is continuous at t_0 over the set $t_0, t_0+h_1, t_0+h_2, \cdots$. This result is used to show that if $x(t)$ and $y(t)$ are measurable over $-\infty < t < \infty$, then the measurable upper bound over $-\infty < t < \infty$ of $|x(t+\lambda)-y(t)|$ is a lower semicontinuous function $G(\lambda)$. In case $y(t) = x(t)$, zeros of $G(\lambda)$ are *essential periods* of $x(t)$. It is shown that if $x(t)$ is measurable, essentially periodic, and not essentially constant, then $x(t)$ has a least positive essential period h and each essential period of $x(t)$ is an integer multiple of h. This implies that if $x(t)$ is measurable, periodic, and not essentially constant, then $x(t)$ has a least positive period H and each period of $x(t)$ is an integer multiple of H. (Received September 22, 1939.)

406. C. B. Allendoerfer: *The Euler number of an imbedded Riemann manifold.*

The space considered is a complete (closed) Riemann manifold R_n of n (even) dimensions imbedded in a euclidean space E_{n+q} where q may have any positive integral value. The total curvature of K of R_n is defined, and it is proved that the integral of K over R_n equals the Euler number of R_n times half the area of an n-sphere. The theorem is well known when $q = 1$. (Received September 27, 1939.)

407. H. A. Arnold: *Postulates for the defective group.*

The postulates for the defective group D are (1) If x and y are distinct elements of D, there is a unique element $z=x\cdot y$ of D, corresponding to the ordered pair x, y. (2) If x, y, z are elements of D, and if $x\cdot y$, $y\cdot z$, $(x\cdot y)\cdot z$, $x\cdot(y\cdot z)$ all exist and are elements of D, then $(x\cdot y)\cdot z=x\cdot(y\cdot z)$. (3) If D has at least one element, then there exists at least one element e (identity) such that $e\cdot b$ exists and $eb=b$ for every element b of D. (4) If e is an element of D with the property in (3), and b is an element of D, there is at least one element x (the inverse of b) in D such that $x\cdot b$ exists and is an element of D and $x\cdot b=e$. The equality is taken to be logical identity. It is proved that $y\cdot e$ is an element of D for all y, that $y\cdot e=y$; that e is unique; that the inverse of an element is unique. A theory of homomorphism and its connection with Galois theory is formulated. The usual strengthening of Postulate (2) results in a Brandt Gruppoid or a Loewy Mischgruppe. A Brandt Gruppoid is homomorphic to a defective group. The strengthened defective groups of lower orders that are not groups are enumerated. (Received October 2, 1939.)

408. L. A. Aroian: *New continued fractions for the incomplete beta function.*

After a brief review of previous methods for the evaluation of the incomplete beta function $B_x(p, q)=\int_0^x x^{p-1}(1-x)^{q-1}dx$, the author shows that J. H. Muller's continued fraction in conjunction with a new continued fraction of the corresponding type furnishes a general method covering the range of the incomplete beta function for all values of x, p, and q. Convergence properties of Muller's continued fraction and the new continued fraction are established. The equivalent and associated types of continued fractions for the new continued fraction with their general properties are also given. While an upper bound is given for the error term in all cases, a remainder to the new continued fraction has not been found. Numerical examples illustrate these methods. Comparisons with previous methods show the superiority of the two continued fractions both from the point of view of theory and practice. (Received September 29, 1939.)

409. L. M. Blumenthal and C. V. Robinson: *A new metric characterization of the straight line.*

In this paper the following new metric characterization of the straight line is obtained: A necessary and sufficient condition that a complete convex externally convex metric space M (containing at least two points) be congruent with the straight line is that M does not possess three points a, b, c such that dist (a, b) = dist (b, c) = dist $(a, c)>0$. An earlier characterization of the straight line, given by Menger (Ergebnisse eines mathematischen Kolloquiums, Vienna, vol. 4 (1932), pp. 41–43) in terms of the notion of *quasi congruence order* is an immediate consequence of the above characterization, the proof of which is much less difficult than that offered for the earlier theorem. It follows from the above result that the simple property "absence of equilateral triples" is logically equivalent to the property "quasi congruence order 3" for complete convex externally convex metric spaces. (Received October 2, 1939.)

410. Salomon Bochner: *Harmonic surfaces in Riemann metric.*

A two-dimensional manifold in a given Riemannian space is a harmonic surface if it is a solution of the system of equations $\partial^2x^i/\partial u^2+\partial^2x^i/\partial v^2+\Gamma^i_{pq}(x)(\partial x^p\partial x^q/\partial u\partial u$

$+\partial x^p \partial x^q/\partial v \partial v) = 0$. The paper presents some general properties of such surfaces and a compactness theorem for families of surfaces. (Received September 27, 1939.)

411. Alfred Brauer: *On the non-existence of the euclidean algorithm in certain quadratic number fields.*

The problem of determining all real euclidean quadratic fields $P(m^{1/2})$ has not been solved completely (cf. Hardy and Wright, *An Introduction to the Theory of Numbers*, 1938, pp. 212–217). Fifteen such fields are known. Except for these 15 fields the euclidean algorithm can only exist in the following cases (p and q denote primes): (1) $m = p \equiv 13 \pmod{24}$; (2) $m = p \equiv 1 \pmod 8$; (3) $m = pq \equiv 1 \pmod{24}$. Moreover, the algorithm does not exist if m is sufficiently large. This was proved by analytical methods by Erdös and Ko for cases 1 and 2 (Journal of the London Mathematical Society, vol. 13 (1938), pp. 3–8), and by Heilbronn for case (3) (Proceedings of the Cambridge Philosophical Society, vol. 34 (1938), pp. 521–526). These results, however, give no bound m_0 such that the fields $P(m^{1/2})$ are not euclidean for $m > m_0$. In this paper it is shown by elementary methods that the algorithm does not exist in case (1) for $p > 109$. In the proof, an inequality for the least odd quadratic nonresidue, obtained by the author some years ago (Mathematische Zeitschrift, vol. 33 (1930), pp. 161–176), is improved. (Received September 29, 1939.)

412. R. P. Dilworth: *Lattices with unique irreducible decompositions.*

The main result of this paper is the following theorem: Let $\mathfrak{S}$ be a lattice with unit element in which every quotient lattice has finite dimensions. Then each element of $\mathfrak{S}$ is uniquely expressible as a reduced crosscut of irreducibles if and only if $\mathfrak{S}$ is a Birkhoff lattice in which every modular sublattice is distributive. (Received September 26, 1939.)

413. R. P. Dilworth: *The arithmetical theory of Birkhoff lattices.*

Let $\mathfrak{S}$ be a Birkhoff lattice with unit element in which every quotient lattice has finite dimensions. Let u_a denote the union of the elements covering a, and $\mathfrak{S}_a$ denote the quotient lattice of elements x such that $u_a \geqq x \geqq a$. An element b of $\mathfrak{S}_a$ is said to be *characteristic* if there exists an irreducible of $\mathfrak{S}$ which divides exactly the same points of $\mathfrak{S}_a$ as b. It is shown that a has a reduced irreducible decomposition $a = [q_1, \cdots, q_k]$ if and only if a has the reduced decomposition $a = [b_1, \cdots, b_k]$ where b_i is a characteristic element of $\mathfrak{S}_a$ divisible by q_i. Characteristic elements are determined in terms of the structure of the lattice as follows: an element b of $\mathfrak{S}_a$ is characteristic if and only if there exists a divisor x of b such that (x, b') covers x and $[x, b'] = a$ for every complement b' of b in $\mathfrak{S}_a$. It is then shown that the number of components in the reduced irreducible decompositions of the elements of $\mathfrak{S}$ is unique if and only if $\mathfrak{S}_a$ is a modular sublattice of $\mathfrak{S}$ for each element a of $\mathfrak{S}$. A number of related arithmetical questions are also treated. (Received September 26, 1939.)

414. J. L. Doob and Warren Ambrose: *On continuous stochastic processes.*

The theory of continuous stochastic processes is the theory of measure on a space of functions of a real variable. One way of establishing such a measure has been given by Paley and Wiener (*Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, pp. 140–178; also see earlier papers by Wiener which are cited in this reference) and another has been

given by Doob (Transactions of this Society, vol. 42 (1937), pp. 107–140). This paper establishes an equivalence between these two theories of measure. (Received October 2, 1939.)

415. Aaron Fialkow: *The conformal theory of curves.*

For a curve C in any Riemann space V_n, $(n>2)$, define an integral invariant S and $(n-1)$ differential invariants $J_1, J_2, \cdots, J_{n-1}$ which remain unchanged by any conformal transformation of V_n. The J's and S are the "conformal curvatures" and "conformal length" respectively of C. The J's (except J_{n-1}) appear in conformal analogues of the ordinary Frenet formulas. If $V_n \longleftrightarrow \overline{V}_n$, $C \longleftrightarrow \overline{C}$ by a conformal map, then the J's are the same functions of S for C and $\overline{C}$. Conversely, if V_n and $\overline{V}_n$ are conform-euclidean spaces and the J's for C and $\overline{C}$ are the same functions of S, then a conformal mapping exists for which $V_n \longleftrightarrow \overline{V}_n$, $C \longleftrightarrow \overline{C}$. If V_n is conform-euclidean, then $J_q \equiv 0$, $(1 \leqq q < n-1)$, if and only if C is conformally equivalent to a curve in an euclidean n-space whose $(q+1)$st metric curvature vanishes. For $n=2$, this theory applies if the conformal mappings are restricted to inversions of spaces of constant curvature. Throughout this paper, systematic use is made of a new kind of tensor differentiation which has conformal meaning. (Received September 28, 1939.)

416. M. C. Foster: *Note on autopolar curves.*

This paper considers autopolar curves as special solutions of differential equations invariant under the dual substitutions, for which the conic of reference is a parabola. (Received September 25, 1939.)

417. H. L. Garabedian: *Theorems associated with the Riesz and the Dirichlet's series methods of summation.*

The following theorems suggested by a theorem due to G. H. Hardy are established. (1) Suppose that $\lim_{n\to\infty} s_n e^{-\nu_n s} = 0$ for all values of s for which $\Re(s) > 0$, that the series $\sum u_n$ is summable $(\lambda, 1)$ to the sum U, and that ν_n is a logarithmico-exponential function of λ_n such that $\nu_n = O(\lambda_n^c)$, where c is any positive constant. Then the series $\sum u_n e^{-\nu_n s}$ converges for $\Re(s) > 0$ and $\lim_{s\to 0} \sum u_n e^{-\nu_n s} = U$. (2) The Dirichlet's series definitions of summability include $(\lambda, 1)$ summability provided that ν_n is a logarithmico-exponential function of λ_n which tends to infinity with n but not as slowly as $\log n$ nor faster than λ_n^c, where c is any positive constant however large. The special case of the second theorem which pertains to logarithmic summability $(\log n, 1)$ is of particular interest. (Received August 15, 1939.)

418. Abe Gelbart: *On functions of two complex variables given by power series expansion.*

Let $f(z_1, z_2) = \sum a_{mn} z_1^m z_2^n$ be an entire function and let $W^2(r)$ be the smallest convex enveloping domain containing the set of points that $f(z_1, z_2)$ assumes on $E[z_2 = Ar^s e^{i\phi},\ z_1 = h(z_2, re^{i\lambda});\ 0 \leqq \phi,\ \lambda \leqq 2\pi;\ r_0 \leqq r < \infty]$. Consider the coefficients of $\exp[f(z_1, z_2)e^{i\alpha}]$. By using a certain generalized Newton polygon, determine a quantity $M(\alpha, r)$ for every α and r. With r constant, construct a line making an angle α with the x axis and having a distance $M(\alpha, r)$ from the origin. $R^2(r)$ is the domain enveloped by these lines, $0 \leqq \alpha \leqq 2\pi$. Then, using the theory of distinguished surfaces, especially the integral formula given by Bergmann (Mathematische Zeitschrift, vol. 39 (1935), pp. 76–94), it is shown that $W^2(r) \supset R^2(r)$. (Received September 28, 1939.)

419. T. S. George and A. J. Maria: *Equilibrium point of Green's function for an n-dimensional spherical shell.*

Let the pole and the corresponding equilibrium point of the Green's function be at distances r_0 and r respectively from the center of the concentric spheres bounding the shell. The authors show that dr/dr_0 is not zero for any position of the pole in the shell and that as the pole moves toward either boundary the equilibrium point remains away from the boundary. (Received September 28, 1939.)

420. P. W. Gilbert: *Two-to-one transformations on acyclic continuous curves.*

It is shown that if M is an acyclic continuous curve which is the sum of a finite number of arcs, there does not exist a continuous 2-1 transformation defined on M. An acyclic curve K is found having the following property: If M is any acyclic continuous curve on which it is possible to define a continuous 2-1 transformation, then M contains K topologically. (Received September 30, 1939.)

421. Nathan Jacobson: *A note on hermitian forms.*

Let Φ_0 be any field of characteristic not equal to 2 and $\Phi=\Phi_0(i)$ or $\Phi_0(i, j)$, respectvely, a quadratic extension or a quaternion division algebra over Φ_0 with $\alpha\rightarrow\bar{\alpha}$, the usual conjugation in these division algebras. To each hermitian matrix A ($\bar{A}'=A$) with elements in Φ associate a symmetric matrix A_1 with elements in Φ_0 and prove that a necessary and sufficient condition that two such matrices be cogredient ($B=\bar{M}'AM$, M nonsingular) is that the corresponding symmetric matrices be cogredient ($B_1=N'A_1N$, N nonsingular). It is thus possible to extend the theory of symmetric matrices over certain special fields (for example, algebraic number fields, p-adic fields) to these two types of hermitian matrices. (Received October 3, 1939.)

422. S. A. Jennings: *The structure of the group ring of a p-group over a modular field.*

A basis for the radical N of the group ring of a p-group over a modular field whose characteristic divides the order of the group consists of all elements of the form $A-E$, where A is any element, and E is the unit element, of the group. By considering the characteristic subgroups K_i of all group elements which may be written in the form $E+n_{i+1}$, where n_{i+1} belongs to N^{i+1}, we get a series $K_0\geqq K_1\geqq\cdots\geqq K_i\geqq\cdots$, terminating with the unit element. We determine a basis for, and the rank of, N^w modulo N^{w+1} in terms of the elements and orders of these subgroups, and identify them abstractly. (Received September 19, 1939.)

423. E. S. Kennedy: *Exponential analogues of the Lambert series.*

The general class of series considered is obtained by replacing z^n in the classical Lambert series with $e^{-\lambda_n x}$, $(0<\lambda_n<\lambda_{n+1}\rightarrow\infty)$. However, the major portion of the paper deals with the "ordinary" series, the special case resulting when $\lambda_n=\log n$. Convergence theorems previously stated by Iyer are proved in detail. It is shown that any series of the general class can be developed as a unique Dirichlet series, and vice versa. In the ordinary case the abelian theorem of Iyer is generalized to include "complex," or Stolz-path, approach. Natural boundary theorems are proved with the aid of a theorem analogous to that of Franel in the Lambert series theory. This theorem is shown to hold for Stolz-path, as well as for "real," approach. Based on the general series, a method of summability is defined which generalizes the "Lambert summability" of Ananda-Rau. (Received September 18, 1939.)

424. Alfred Korzybski: *General semantics: extensionalization in mathematics, mathematical physics, and general education.* III. *Over/under defined terms.*

This paper, the third of its series, shows that even in mathematics, physics, and life we deal only with over/under defined terms, resulting in extreme difficulties of thinking. Thinking$_1$ of apes, insane, and so on, differs from thinking$_2$ of normal humans. Formalism$_1$ of insane, and so on, contrasts with normal formalism$_2$, including mathematics. Calculus of propositions requires formalism$_2$, needed for sanity. The author shows mathematics as a language similar in structure to imperical facts, and the structure of the nervous system, allowing predictability in science and life. Undefined terms and postulates solve the dilemma of knowing everything before we know anything. The multiordinality of knowledge is explained. Plain human ignorance is often impossible, only false knowledge exists, inducing insanity. Ignorance is static in character; false knowledge is dynamic. Physico-mathematical methods uniquely translate static into dynamic, and vice versa. Evaluation by intensional methods, or mere verbal definition, induces dementia praecox. It is explained how physico-mathematical, extensional methods are essential for sanity. The two methods prove irreconcilable. Analysis shows the inherent dualism of over/under defined terms, depending on whether intensional or extensional methods are used, eliminating conflicting dualism in science and semantic (evaluational) reactions. (Received September 14, 1939.)

425. W. T. Martin: *On a minimal problem in the theory of analytic functions of several variables.*

In 1932 Wirtinger proposed and solved, in explicit form, the following problem. Given a region G in the complex z-plane and a function $\phi(z, \bar{z})$ once continuously differentiable in G, to find an analytic function $f(z)$ for which $\int_G |\phi - f|^2 \cdot d\omega = \text{min.}$, where $d\omega$ is the element of area (Monatshefte für Mathematik und Physik, vol. 39 (1932), pp. 377–384). This year he stated the analogous question for several variables and obtained a unique and explicit solution in the case in which the region under consideration is a hypersphere, and ϕ is merely integrable (Monatshefte für Mathematik und Physik, vol. 47 (1939), pp. 426–431). He conjectured the existence of a solution for general regions. Using the theory of the kernel of a region for functions of several complex variables (Bergmann, Mathematische Zeitschrift, vol. 29 (1929), pp. 640–677), the author proves Wirtinger's conjecture for a wide class of regions which includes all bounded regions and for $\phi \in L^2$ over the region. Certain extensions of the problem are also considered, for example, where f is required, in addition, to take on given values at given points. The solution is again unique and explicit. (Received September 25, 1939.)

426. Deane Montgomery and Leo Zippin: *Note on rotation-group of the two-sphere.*

Let S denote the unit two-sphere (in ordinary space) and G the familiar rotation-group of S. This note shows by elementary topological considerations that G has no subgroup (itself excepted) which is transitive on S. The subgroups considered are of entirely arbitrary nature. This is of interest in connection with the problem of "rigid geometries" in 3-space (see abstract 45-11-402 by these authors). (Received September 28, 1939.)

427. D. C. Murdoch: *Further theory of quasi-groups.*

An "abelian" quasi-group has been previously defined by the author as one which satisfies the law $(ab)(cd)=(ac)(bd)$. The structure of these quasi-groups is now studied from the point of view of extension theory. There are two types of extensions: those which preserve the right-unit quasi-group and those which enlarge it. Every abelian quasi-group can be obtained from a "self-unit" quasi-group (one which is identical with its own right-unit quasi-group), by a series of extensions by quotient quasi-groups which satisfy certain restrictions. Some properties of self-unit quasi-groups are discussed and the Shreier conditions for the extensions are obtained. Certain non-abelian quasi-groups are also treated, namely those satisfying the associative law $a(bc)=(a^sb)c^a$, where s and t are automorphisms and $st=ts$. (Received September 27, 1939.)

428. Rufus Oldenburger: *Exponent trajectories in symbolic dynamics.*

In treating symbolic trajectories, Marston Morse has used exponents on elements. In the present paper, it is proved that the exponents on the elements in a recurrent trajectory T with two or more generating symbols form a unique recurrent trajectory T_e. The trajectory T_e is termed the exponent trajectory of T. If T is periodic, then T_e is periodic, and if T contains exactly two generating symbols, T is periodic if and only if T_e is periodic. If T is periodic, $T\neq T_e$. There exist nonperiodic trajectories T such that $T=T_e$, and if T contains the generating symbols 1, 2 only, there is one and only one such trajectory T. The introduction of the notion of exponent trajectory gives a new method of constructing recurrent trajectories. Certain methods of constructing recurrent trajectories from a given recurrent trajectory have been found by Morse and Hedlund. In a manner analogous to the treatment of exponent trajectories, it is also proved in the present paper that the exponents on the elements in a transitive ray form a unique transitive ray. (Received September 28, 1939.)

429. Walter Prenowitz: *Projective geometry as a group-like system.*

Garrett Birkhoff (Annals of Mathematics, (2), vol. 36 (1935), pp. 743–748) gives a lattice theoretic treatment of projective geometry taking as undefined the set S of all linear subspaces of a projective space and the operations meet and join in S. In this paper a less abstract treatment is given. The primitive terms are point and the operation join as applied to a pair of points. The postulates characterize a projective space as a type of "group" (closely related to the so-called multi-group) in which join, the operation of composition, is many-valued. This leads to a group theoretic development of projective geometry in which linear subspaces appear in the guise of subgroups. The properties of meet and join of linear subspaces in n-dimensional projective geometry are studied by algebraic methods, the austauschsatz of modern algebra playing an important role in the treatment. (Received September 20, 1939.)

430. J. F. Randolph: *Metric separability.*

The results of this paper were announced in this Bulletin, abstract 43-1-50, but the methods of proof are considerably improved. (Received October 2, 1939.)

431. C. J. Rees: *Differential equation of elliptic orthogonal polynomials.*

Following the method of Shohat (Duke Mathematical Journal, vol. 5 (1939)),

a differential equation is derived for elliptic orthogonal polynomials. As is known from the general theory, it is a linear homogeneous differential equation of the second order. Its coefficients are given in explicit form and involve one unknown parameter only, for which a recurrence relation is given. An application of this differential equation is made to the study of roots of the elliptic orthogonal polynomial. (Received September 14, 1939.)

432. H. J. Riblet: *Symmetric differential expressions and applications.*

This paper introduces a theory for symmetric differential expressions comparable to that for ordinary symmetric functions. It is shown that every integral rational differential function of n quantities $y_1, \cdots, y_n$ which is of first order and which has coefficients in a commutative field R_1 can be expressed as a rational integral function of the elementary symmetric functions of the n quantities and their first derivatives with coefficients in R_1. The identity $S_m^* + a_1 S_{m-1}^* + \cdots + a_{m-1} S_1^* + a_m' = 0$ is established, wherein $S_i^* = \sum x_i^{i-1} x_i'$ and the primes denote differentiation. Applications are made to Waring's formula for the power sums and to solutions of total differential equations. (Received October 5, 1939.)

433. J. H. Roberts: *A theorem on two-to-one transformations.*

O. G. Harrold has shown (this Bulletin, abstract 45-1-20) that there does not exist a continuous 2–1 transformation of an arc into a circle. A transformation is said to be 2–1 if every image point has exactly 2 inverse images. In the present paper it is shown that it is not possible to define a continuous 2–1 transformation on a closed 2-cell. The author has not been able to generalize this result to the closed n-cell. (Received September 30, 1939.)

434. Benjamin Rosenbaum: *Ideals of a quadratic field in canonical form.*

The canonical forms of the product, quotient (when it exists), greatest common divisor, and least common multiple of two ideals of a quadratic field are obtained from the canonical forms of the two ideals. (A definition of the canonical form of an ideal in a quadratic field can be found in one of the books on algebraic numbers by Hancock, Reid or Sommer.) Also, new results are obtained in the determination of necessary and sufficient conditions that an ideal of a quadratic field be a principal ideal. (Received September 11, 1939.)

435. W. T. Scott and H. S. Wall: *A convergence theorem for continued fractions.* III.

If the inequalities in the convergence theorem announced in abstracts 45-7-277 and 45-7-285, this Bulletin, are satisfied by $r_1, r_2, r_3, \cdots$ which are real and positive and such that the products $r_1 r_3 r_5 \cdots r_{2n-1}$ and $r_2 r_4 r_6 \cdots r_{2n}$ are bounded, with actual inequality for at least one odd value of n and for at least one even value of n, then the continued fraction converges if (and only if) the series $\sum |b_n|$ diverges, where $a_1 = 1/b_1 = 1$, $a_n = 1/b_n b_{n-1}$, $(n = 2, 3, 4, \cdots)$. In particular, if the a_n lie within or upon the boundary of the parabola $|z| - R(z) = 1/2$, then the continued fraction converges if and only if the series $\sum |b_n|$ diverges. With the aid of this convergence theorem, it is shown that the secondary requirement of the divergence of the series $\sum (p_1 - 1)(p_2 - 1) \cdots (p_n - 1)$ in Pringsheim's uniform convergence theorem (Perron, p. 262, Theorem 30) is not essential. (Received September 12, 1939.)

436. D. T. Sigley: *k-set groups.*

A k-set group is defined as a finite abstract group which contains precisely k complete sets of non-invariant conjugate subgroups. The order of a k-set group is divisible by not more than $k+1$ distinct prime factors. There exists at least one k-set group for every integral value of k. This paper includes the determination of all 1-set and 2-set groups. (Received August 25, 1939.)

437. A. W. Tucker: *The algebraic structure of complexes.* Preliminary report.

Let $x_1, x_2, \cdots, x_n$ denote the cells (of all dimensions) of a complex X. Let E and F be square matrices of order n which are such that $\sum x_i E_i^j = (-1)^p x_i$, where $p = \dim x_i$, and $\sum x_j F_i^j$ is the boundary chain of x_i. Then (1) $EE=1$, (2) $FF=0$, and (3) $EF+FE=0$. Conversely, any two matrices E and F which satisfy these three conditions give rise to an algebraic system which behaves in many respects like a complex. Included in the part of the theory of complexes which is characterized in this simple algebraic fashion is the theory of chain-mappings, their coincidences and graphs, as well as a general theory of products. (Received October 2, 1939.)

438. W. J. Trjitzinsky: *Some problems in the theory of singular integral equations.*

An extensive and systematic treatment of integral equations of any finite or transfinite rank has been given recently by Trjitzinsky (Transactions of this Society, vol. 41 (1939), pp. 202–279). In this connection, Carleman's kernels, which constitute a very important extension of the better known classical kernels, were designated as of rank one. The kernels of rank $\alpha>1$ are limits, in a suitable sense, of kernels of lower ranks. Thus there arose a transfinite hierarchy of theories of ascending generality. In Carleman's and in Trjitzinsky's work emphasis is on results for non-real values of the parameter. Now, both from the point of view of mathematical interest and from the point of view of important applications to mathematical physics, developments for real values of the parameter are essential. Such developments, including a treatment of equations of the first kind, are extensively given in the present paper, which will appear in the Annals of Mathematics. (Received September 30, 1939.)

439. A. H. Wheeler: *A one-sided polyhedron having one hundred twenty faces.*

This paper deals with some of the properties of a one-sided polyhedron which has 120 faces, 90 vertices and 210 edges satisfying the relation $e-k+f=0$. Thirty of its edges can be placed in coincidence with the 30 edges of a regular icosahedron, and when so placed it surrounds and lies entirely outside of the icosahedron. It is a member of a family of one-sided polyhedra having $6n$ faces, where $n \geqq 2$. (Received September 13, 1939.)

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